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326. ON AN INEQUALITY*

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Theorem. *If* $a_1, \dots, a_n$ *are real positive numbers, and if* $p_1, \dots, p_n$ *are real nonnegative numbers, and if* $b>1$, *then*

$$(1) \qquad \sum_{k=1}^{n} b^{\left(a_i^{\sum_{k=1}^{n} p_k}\right)} \geqq \frac{1}{(n-1)!} \sum_{p(n)} b^{\left(a_1^{p_{i_1}} \cdots a_n^{p_{i_n}}\right)},$$

where $\sum_{p(n)}$ *denotes the summation over all permutations* $i_1, \dots, i_n$ *of the set* $\{1, \dots, n\}$.

Equality holds in (1) *if and only if* $a_1 = \cdots = a_n$ *or if all* p_i *are equal to zero, except perhaps one of them.*

Proof. Let us first prove the inequality

$$(2) \qquad b^{x_1} + b^{x_2} \geqq b^{y_1} + b^{y_2},$$

where $b>1,\ x_i \geqq 0,\ y_i \geqq 0,\ x_1+x_2 \geqq y_1+y_2,\ \max_i (x_i) \geqq \max_i (y_i) \quad (i=1, 2).$

In the opposite case we would have

$$b^{x_1} + b^{x_2} < b^{y_1} + b^{y_2},$$

i. e., supposing that $x_2 \geqq x_1$,

$$b^{x_1+x_2} + b^{2x_2} < b^{y_1+x_2} + b^{y_2+x_2},$$

and thus

$$b^{y_1+y_2} + b^{2x_2} < b^{y_1+x_2} + b^{y_2+x_2},$$

i.e.,

$$(b^{x_2} - b^{y_2})(b^{x_2} - b^{y_1}) < 0$$

which is absurd, since $x_2 \geqq \max(y_1, y_2)$ and $b>1$. This proves inequality (2).

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As the inequalities

$$a_1^{p_1+p_2}+a_2^{p_1+p_2}\geqq a_1^{p_1}a_2^{p_2}+a_1^{p_2}a_2^{p_1},$$

$$\max_k(a_k^{p_1+p_2})\geqq\max(a_1^{p_1}a_2^{p_2},\ a_1^{p_2}a_2^{p_1}),\qquad (k=1,2)$$

can be easily proved we can put in (2)

$$x_i=a_i^{p_1+p_2},\quad y_i=a_1^{p_i}a_2^{p_j},\qquad (i,j=1,2;\ i\neq j),$$

which implies (1) for $n=2$.

Suppose that (1) holds for a natural number $n>2$, and let us prove that it holds for $n+1$.

Putting $q_s=p_s$ $(s=1,\ldots,n-1)$, $q_n=p_n+p_{n+1}$, then, by the hypothesis we have

$$\sum_{j=1}^{n} b^{\left(a_{i_j}^{\sum_{k=1}^{n} q_k}\right)}\geqq\frac{1}{(n-1)!}\sum_{q(n)} b^{\left(a_{i_1}^{q_{k_1}}\cdots a_{i_n}^{q_{k_n}}\right)},\tag{3}$$

where $i_j=1,\ldots,n+1$; $i=1,\ldots,n+1$; $i_j\neq i$; $i_1<i_2<\cdots<i_n$, and $\sum_{q(n)}$ has the same meaning as before $\{1,\ldots,n\}$.

Adding all the $n+1$ inequalities (3) we find

$$\sum_{i=1}^{n+1} b^{\left(a_i^{\sum_{k=1}^{n} q_k}\right)}\geqq\frac{1}{n!}\sum_{i=1}^{n+1}\sum_{q(n)} b^{\left(a_{i_1}^{q_{k_1}}\cdots a_{i_n}^{q_{k_n}}\right)}.\tag{4}$$

On the right hand side in (4) we have $(n+1)!$ summands; consider one of them

$$X=b^{\left(a_{i_1}^{q_{k_1}}\cdots a_{i_r}^{q_{k_r}}\cdots a_{i_n}^{q_{k_n}}\right)},$$

and let $q_{k_s}=p_{k_s}$, $q_{k_r}=p_n+p_{n+1}$ $\qquad(k_s,\ k_r=1,\ldots,n;\ k_s\neq k_r)$.

Then clearly there is one and only one summand Y on the right in (4) whose representation is different from the represantation of X only in that a_{i_r} is replaced by a_i, where $\{i\}=\{\{1,\ldots,n+1\}\setminus\{i_1,\ldots,i_n\}\}$;denotes their common part by B; since inequality (1) was proved for $n=2$, the sum of the above summands X and Y can be minorized by

$$B^{\left(a_{i_r}^{p_n+p_{n+1}}\right)}+B^{\left(a_i^{p_n+p_{n+1}}\right)}\geqq B^{\left(a_{i_r}^{p_n}a_i^{p_{n+1}}\right)}+B^{\left(a_{i_r}^{p_{n+1}}a_i^{p_p}\right)},\tag{5'}$$

i. e.,

$$X+Y\geqq b^{\left(a_1^{p_{k_1}}\cdots a_{n+1}^{p_{k_{n+1}}}\right)}+b^{\left(a_1^{p_{s_1}}\cdots a_{n+1}^{p_{s_{n+1}}}\right)}.\tag{5}$$

But, $(k_1,\ldots,k_{n+1})$ and $(s_1,\ldots,s_{n+1})$ are two different permutations of the set $\{1,\ldots,n+1\}$, (actually, they differ in that n and $n+1$ have changed places).

If among all the summands on right of (4) we pair off the corresponding ones, like X and Y, and use (5), we shall obtain $(n+1)!$ summands of the form

$$b^{\left(a_1^{p_{k_1}} \cdots a_{n+1}^{p_{k_{n+1}}}\right)},$$

where $(k_1, \ldots, k_{n+1})$ is the permutation of the set $\{1, \ldots, n+1\}$.

From the construction of these permutations and the fact that among the permutations $q(n)$ there no equal one, it follows that all the permutations are mutually different, and since there are $(n+1)!$ of them, it means that exhaust the set of all permutations of $\{1, \ldots, n+1\}$.

This fact, together with (4), yields (1) for $n+1$.

We immediately conclude that equality holds in (1) if $a_1 = \cdots = a_n$ or $p_1 = \cdots = p_n = 0$. Let us now prove that equality can also hold only in the case when one the p_i's is not zero.

First, from the proof of (1) for $n=2$ we deduce that the above assertion holds for $n=2$. Suppose that it is true for some $n>2$, and let us prove that it holds for $n+1$.

Without loss of generality we can suppose that $p_n \neq 0$; namely, in the construction of (3) we could have taken $q_n = p_i + p_k$ $(i, k = 1, \ldots, n+1;\ i \neq k)$, and not $q_n = p_n + p_{n+1}$ as we have done for symmetry's sake.

Therefore, all the q_i's except q_n are equal to zero, and owing to the inductive hypothesis equality holds in (3) for all $i \in \{1, \ldots, n+1\}$; which means that equality will hold also in (4) under the above conditions.

Furthermore, suppose that not all a_i's are equal; otherwise equality would hold directly in (1). Suppose that two of them are not equal.

Consider on the right hand side of (4) those summands X and Y so a_{i_r} and a_i whose exponents in X and Y are $q_n = p_n + p_{n+1}$ are the mentioned pair of a_i's. Clearly they are uniquely determined.

As we have proved that equality holds in (4), in order that it holds in (1) for $n+1$, it must hold in (5′), i.e., in (5).

Howerer, (5′) coincides with (1) for $n=2$, and equality will hold in (5′) if and only if $a_{i_r} = a_i$ or if at least one of p_n and p_{n+1} is equal to zero. By the hypothesis, $a_{i_r} \neq a_i$ and $p_n \neq 0$, which means that $p_{n+1} = 0$.

This completes the proof.

Remark 1. Analogously we can prove the somewhat simpler inequality

$$\text{(1′)} \qquad \sum_{i=1}^{n} a_i^{\sum_{k=1}^{n} p_k} \geqq \frac{1}{(n-1)!} \sum_{p(n)} a_1^{p_{k_1}} \cdots a_n^{p_{k_n}},$$

under the some conditions which where supposed in (1).

EXAMPLE 1. Putting in (1′) $p_k = 1$ $(k = 1, \ldots, n)$, we obtain the arithmetic-geometric mean inequality for n positive numbers

$$\frac{1}{n}(a_1 + \cdots + a_n) \geqq (a_1 \cdots a_n)^{\frac{1}{n}}.$$

EXAMPLE 2. Since (1′) holds for every $p_i \geqq 0$ $(i=1, \ldots, n)$, putting $p_i = kq_i$ $(i=1, \ldots, n;$ $k=0, 1, \ldots)$ and (1′) in summing these inequalities with respect to k, with $0<a_i<1$ $(i=1, \ldots, n)$, we get

$$\sum_{i=1}^{n} \frac{1}{1-a_i^{\sum_{k=1}^{n} q_k}} \geqq \frac{1}{(n-1)!} \sum_{q(n)} \frac{1}{1-a_1^{q_{i_1}} \cdots a_n^{q_{i_n}}}.$$

Remark 2. Putting in (1) $p_{k+1} = \cdots = p_n = 0$ $(1 \leqq k \leqq n)$, then

$$\sum_{i=1}^{n} b\Big(a_i^{\sum_{j=1}^{k} p_j}\Big) \geqq \frac{(n-k)!}{(n-1)!} \sum_{C(k)} \sum_{p(k)} b\big(a_{c_1}^{p_{i_1}} \cdots a_{c_k}^{p_{i_k}}\big)$$

where $\sum_{C(k)}$ denotes the sum over all the combinations $(c_1, \ldots, c_k)$ of k-th order of the set $\{1, \ldots, n\}$.

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