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325. ON AN INEQUALITY INVOLVING SYMMETRIC FUNCTIONS*

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Inequality (1) was proposed in 1968 at a Student's Competition in Hungary. Professor D. S. MITRINOVIĆ has drawn my attention to it and has suggested to me to generalize it. Realization of his advice is the subject of this paper.

For any nonnegative numbers $a_1, \ldots, a_n$ we have

(1) $$(n-1)\sum_{i=1}^{n} a_i^n + n\prod_{i=1}^{n} a_i \geqslant \sum_{i=1}^{n} a_i^{n-1}\sum_{i=1}^{n} a_i .$$

In order to prove (1), we shall find it convenient to introduce the following notations:

$$S_r^n=\sum_{i=1}^{r} a_i^n, \qquad S_{r,k}^n=\sum_{\substack{i=1\\ i\neq k}}^{r} a_i^n,$$

$$P_r=\prod_{i=1}^{r} a_i, \qquad P_{r,k}=\prod_{\substack{i=1\\ i\neq k}}^{r} a_i,$$

and to prove the following Lemma:

For all natural numbers r and n we hawe

(2) $$r(r-1)S_r^{n+1}+S_r^{n-1}(S_r)^2\geqq 2(r-1)S_r^n S_r+S_r^{n-1} S_r^2.$$

***Proof of* (2).** Suppose that (2) holds for any r nonnegative numbers $a_1, \ldots, a_r$ and for every natural number n. Then, (2) also holds for r nonnegative numbers $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_r, a_{r+1}$ and for every n, i. e., we have

(3) $$r(r-1)S_{r+1,k}^{n+1}+S_{r+1,k}^{n-1}(S_{r+1,k})^2\geqq 2(r-1)S_{r+1,k}^n S_{r+1,k}+S_{r+1,k}^{n-1} S_{r+1,k}^2$$

for $k=1, \ldots, r+1$.

Since

$$\sum_{k=1}^{r+1} S_{r+1,k}^{n+1}=\sum_{k=1}^{r+1}\left(S_{r+1}^{n+1}-a_k^{n+1}\right)=rS_{r+1}^{n+1},$$

$$\sum_{k=1}^{r+1} S_{r+1,k}^{n-1}(S_{r+1,k})^2=\sum_{k=1}^{r+1}(S_{r+1}^{n-1}-a_k^{n-1})(S_{r+1}-a_k)^2$$

$$=(r-2)S_{r+1}^{n-1}(S_{r+1})^2+2S_{r+1}^n S_{r+1}+S_{r+1}^{n-1} S_{r+1}^2-S_{r+1}^{n+1},$$

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$$\sum_{k=1}^{r+1} S_{r+1,k}^{n-j+1} S_{r+1,k}^{j} = \sum_{k=1}^{r+1} (S_{r+1}^{n-j+1} - a_k{}^{n-j+1})(S_{r+1}^{j} - a_k{}^{j})$$

$$= (r-1) S_{r+1}^{n-j+1} S_{r+1}^{j} + S_{r+1}^{n+1} \quad \text{for } j = 1, \ldots, n,$$

adding inequalities (3) for $k=1, \ldots, r, r+1$, we get

$$(r-2)(r+1) r S_{r+1}^{n+1} + (r-2) S_{r+1}^{n-1} (S_{r+1})^2 \geqq 2(r-2) r S_{r+1}^{n} S_{r+1} + (r-2) S_{r+1}^{n-1} S_{r+1}^{2}$$

i. e., inequality (2) holds for $r+1$ if it holds for some $r>2$.

For $r=1$ or $r=2$, inequality (2) is true, as it reduces to an equality. For $r=3$ it is equivalent to

$$R = a_1{}^{n-1}(a_1-a_2)(a_1-a_3) + a_2{}^{n-1}(a_2-a_1)(a_2-a_3) + a_3{}^{n-1}(a_3-a_1)(a_3-a_2) \geqq 0$$

which is SCHUR's inequality (see, for example [1], pp. 119—121).

***Proof of* (1).** Suppose that inequality holds for any n numbers $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n, a_{n+1}$, i. e., that the following inequalities hold

$$(4k) \qquad (n-1) S_{n+1,k}^{n} + n P_{n+1,k} \geqq S_{n+1,k}^{n-1} S_{n+1,k} \qquad (k=1, \ldots, n+1).$$

Multiply $(4k)$ by a_k and add. We get

$$(n-1) \sum_{k=1}^{n+1} a_k S_{n+1,k}^{n} + n \sum_{k=1}^{n+1} P_{n+1} \geqq \sum_{k=1}^{n+1} a_k S_{n+1,k}^{n-1} S_{n+1,k},$$

which is equivalent to

$$(5) \qquad (n-1) S_{n+1}^{n} S_{n+1} - (n-1) S_{n+1}^{n+1} + n(n+1) P_{n+1}$$

$$\geqq S_{n+1}^{n-1} (S_{n+1})^2 - S_{n+1}^{n-1} S_{n+1}^{2} - S_{n+1}^{n} S_{n+1} + S_{n+1}^{n+1}.$$

From inequality (5) we obtain

$$(6) \qquad n^2 S_{n+1}^{n+1} + n(n+1) P_{n+1} \geqq n(n+1) S_{n+1}^{n+1} + S_{n+1}^{n-1} (S_{n+1})^2$$

$$- S_{n+1}^{n-1} S_{n+1}^{2} - n S_{n+1}^{n} S_{n+1}.$$

However, inequalities (6) and (2) for the case $r=n+1$, yield

$$n S_{n+1}^{n+1} + (n+1) P_{n+1} \geqq S_{n+1}^{n} S_{n+1},$$

which completes the inductive proof of inequality (1).

Notice that equality holds in (1) if and only if $n \leqq 2$ or $a_1 = \cdots = a_n$ for $n \geqq 3$.

Generalization. *For any r nonnegative numbers $a_1, \ldots, a_r$ and for every natural number $n \leqq r$, the following inequality holds*

$$(7) \qquad (r-1) S_r^n + \frac{r}{\binom{r}{n}} \sigma_r^n \geqq S_r^{n-1} S_r$$

where

$$\sigma_r^n = \sum_{ij \in N_r} \prod_{j=1}^{n} a_{ij}, \qquad N_r = \{1, 2, \ldots, r\}.$$

Again we shall first prove the following:

If $0 < a \leqq x$, *then*

$$(S_r^{x/2})^2 \leqq S_r^{x-a} S_r^a \leqq r S_r^x. \tag{8}$$

A special case of (8) will be used in the proof of (7).

Proof of **(8).** Suppose that (8) holds for any r nonnegative numbers a_i. Then we have

$$(S_{r+1,k}^{x/2})^2 \leqq S_{r+1,h}^{x-a} S_{r+1,k}^a \leqq r S_{r+1,k}^x, \qquad k = 1, \ldots, r+1 \tag{9}$$

$$\sum_{k=1}^{r+1} (S_{r+1,k}^{x/2})^2 \leqq \sum_{k=1}^{r+1} S_{r+1,k}^{x-a} S_{r+1,k}^a \leqq r \sum_{k=1}^{r+1} S_{r+1,k}^x$$

$$(r-1)(S_{r+1}^{x/2})^2 + S_{r+1}^x \leqq (r-1) S_{r+1}^{x-a} S_{r+1}^a + S_{r+1}^x \leqq r^2 S_{r+1}^x$$

$$(r-1)(S_{r+1}^{x/2})^2 \leqq (r-1) S_{r+1}^{x-a} S_{r+1}^a \leqq (r^2-1) S_{r+1}^x$$

where each of the above inequalities implies the next. This sequence of implications is obtained in the following order: 1) summation of inequalities (9) for $k = 1, \ldots, r+1$; using the formula

$$\sum_{k=1}^{r+1} S_{r+1,k}^u S_{r+1,k}^v = (r-1) S_{r+1}^u S_{r+1}^v + S_{r+1}^{u+v}$$

with 2) $u = v = x/2$, 3) $u = x - a$, $v = a$, 4) $u = x$, $v = 0$ $(S_r^0 = r)$.

From the last two inequalities it follows that (8) is true for $r+1$ numbers a_i if it is true for any $r > 1$ numbers $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{r+1}$.

For $r = 1$, inequalities (8) are true, as they reduce to equalities.

For $r = 2$, we get from (8)

$$(a_1^{x/2} + a_2^{x/2})^2 \leqq (a_1^{x-a} + a_2^{x-a})(a_1^a + a_2^a) \leqq 2(a_1^x + a_2^x),$$

which is equivalent to

$$(a_1^{(x-a)/2} a_2^{a/2} - a_1^{a/2} a_2^{(x-a)/2})^2 \geqq 0 \text{ and } (a_1^{x/2} - a_2^{x/2})^2 \geqq 0,$$

which means that inequalities (8) are true for $r = 2$. This concludes the inductive proof of (8). Equality holds there if and only if $r = 1$, or $a_1 = a_2 = \cdots = a_n$ for $n \geqq 2$.

Proof of **(7).** Suppose that (7) holds for any r nonnegative numbers a_i. Then we have

$$(r-1) S_{r+1,k}^n + \frac{r}{\binom{r}{n}} \sigma_{r+1,k}^n \geqq S_{r+1,k}^{n-1} S_{r+1,k} \text{ for } k = 1, \ldots, r+1, \tag{10}$$

where

$$\sigma^n_{r+1,\,k} = \sum_{i_j \in N_{r+1,\,k}} \prod_{j=1}^{n} a_{i_j} \qquad N_{r+1,\,k} = N_{r+1} \setminus \{k\}.$$

Adding inequalities (10) for $k=1, \ldots, r+1$, and taking into account equalities

$$\sum_{k=1}^{r+1} S^n_{r+1,\,k} = rS^n_{r+1},$$

$$\sum_{k=1}^{r+1} S^{n-1}_{r+1,\,k}\, S_{r+1,\,k} = (r-1) S^{n-1}_{r+1}\, S_{r+1} + S^n_{r+1},$$

$$\sum_{k=1}^{r+1} \sigma^n_{r+1,\,k} = (r+1-n)\,\sigma^n_{r+1},$$

we get

$$r^2 S^n_{r+1} + \frac{r(r+1)}{\binom{r+1}{n}}\,\sigma^n_{r+1} \geqq (r-1) S^{n-1}_{r+1}\, S_{r+1} + (r+1) S^n_{r+1}. \tag{11}$$

Inequality (11) and the second inequality of (8) for $x=n$, $\alpha=1$ together yield

$$rS^n_{r+1} + \frac{r+1}{\binom{r+1}{n}}\,\sigma^n_{r+1} \geqq S^{n-1}_{r+1}\, S_{r+1},$$

which means that (7) holds for $r+1$ numbers a_i if it holds for any r numbers a_i.

Applying now the same method as in the proof of (1), we see that this induction proof is also completed.

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REFERENCE

1. D. S. Mitrinović: *Analytic Inequalities*. Grundlehren der Mathematischen Wissenschaften Bd. 165, Berlin-Heidelberg-New York 1970.

COMMENT OF THE REDACTION COMMITTEE

It would be interesting to connect inequality [1] with the following inequality of G. Kober:

$$(n-1) \sum_{i=1}^{n} a_i^n + n \prod_{i=1}^{n} a_i \geqq 2 \sum_{1 \leqq i \leqq j \leqq n} (a_i a_j)^{n/2} + \sum_{i=1}^{n} a_i^n \qquad (a_i \geqq 0,\ n \geqq 2).$$

For this inequality see H. Kober: Proc. Amer. Math. Soc. 9 (1958), 452—459, or [1], pp. 379—380, Section 3. 9. 70.