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308. FROM THE HISTORY OF NONANALYTIC FUNCTIONS, II*

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In paper [1] we have given a short survey of the history of nonanalytic functions, whose aim was partly to establish the place of G. V. KOLOSOV in that theory. We have now found it necessary to publish this addition to [1].

1. We have not stated in [1] that [2] presents KOLOSOV's doctoral dissertation and that [3] is its shortened version. These facts, as well as other information regarding KOLOSOV, can be found in paper [4] of his pupil N. MUSHELIŠVILI.

2. As far as we know it is very difficult to get paper [5], while it seems almost impossible to get [2]. We shall, therefore, expose here KOLOSOV's method of integration of systems of partial differential equations. It can be summarized as follows:

Let

$$F_k(x, y, u, v, u_x, u_y, v_x, v_y)=0 \qquad (k=1,2) \tag{1}$$

be a given system of partial differential equations, where $u=u(x, y)$ and $v=v(x, y)$ are unknown functions, and x and y independent variables. Multiply one of the above equations by i, say the equation $F_2=0$, and add it to the first. We get

$$F_1+iF_2=0.$$

Using the relations

$$2x=z+\bar{z},\ 2iy=z-\bar{z},\ 2u=w+\bar{w},\ 2iv=w-\bar{w},$$

$$Dw=u_x-v_y+i(v_x+u_y),\quad \bar{D}w=u_x+v_y+i(v_x-u_y),$$

and, therefore,

$$4u_x=\bar{D}w+D\bar{w}+Dw+\bar{D}\bar{w},\quad 4u_y=i(\bar{D}w-D\bar{w}-Dw+\bar{D}\bar{w}),$$

$$4v_x=-i(\bar{D}w-D\bar{w}+Dw-\bar{D}\bar{w}),\quad 4v_y=\bar{D}w+D\bar{w}-Dw-\bar{D}\bar{w},$$

we find

$$F_1+iF_2=F(z, \bar{z}, w, \bar{w}, Dw, D\bar{w}, \bar{D}w, \bar{D}\bar{w}).$$

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If F does not depend on $\bar{D}w$, $\bar{D}\bar{w}$, $D\bar{w}$ and $\bar{w}$, KOLOSOV calls equations (1) conjugate equations.

Let equations (1) be conjugate. We then have

$$F_1 + iF_2 = F(z, \bar{z}, w, Dw) = 0. \tag{2}$$

Instead of equation (2) KOLOSOV considers the ordinary differential equation

$$F(a, 2x, y, y') = 0, \tag{3}$$

where a is a parameter.

Let its general solution be given by

$$y = \Phi(x, C),$$

where C is an arbitrary constant. Then

$$w = \Phi\left(\frac{\bar{z}}{2}, f(z)\right), \tag{4}$$

where f is an arbitrary analytic function, is the solution of (2).

Separating the real and imaginary parts of the solution (4) we obtain a solution of system (1) in the form

$$u(x, y) = \operatorname{Re} \Phi\left(\frac{\bar{z}}{2}, f(z)\right),$$

$$v(x, y) = \operatorname{Im} \Phi\left(\frac{\bar{z}}{2}, f(z)\right).$$

Clearly this method can be applied to systems of higher order.

G. V. KOLOSOV did not describe his method in this general way, but rather he explained it on the following examples:

1) System

$$u_x - v_y = x, \qquad u_y + v_x = x - y \tag{5}$$

can be written in the form

$$Dw = \bar{z} + \frac{i}{2}(z + \bar{z}).$$

Its solution is

$$w = \frac{1}{2}\left(\frac{\bar{z}^2}{2}\left(1 + \frac{i}{2}\right) + \frac{i}{2} z\bar{z}\right) + A(z),$$

where A is an arbitrary analytic function. Therefore, a solution of system (5) is given by

$$u(x, y) = \frac{1}{4}(x^2 - y^2 + xy) + f(x, y)$$

$$v(x, y) = \frac{1}{8}(3x^2 + y^2 - xy) + g(x, y),$$

where $A(z) = f(x, y) + ig(x, y)$.

2) System

$$u_x - v_y = \sum_j (k_j P_j + l_j Q_j), \quad u_y + v_x = \sum_j (m_j P_j + n_j Q_j),$$

where k_j, l_j, m_j, n_j are constants and $P_j + iQ_j = f_j(z)$ is an analytic function, can be written in the form

$$Dw = \sum_j [\varepsilon_j f_j(z) + \eta_j g_j(\bar{z})],$$

where $2\varepsilon_j = k_j + l_j + i(m_j + n_j)$, $2\eta_j = k_j - l_j + i(m_j - n_j)$, and $g_j(\bar{z}) = P_j - iQ_j$. Therefore,

$$w = u + iv = \frac{\bar{z}}{2} \sum_j \varepsilon_j f_j(z) + \frac{1}{2} \sum_j \eta_j \int g_j(\bar{z})\, d\bar{z} + A(z)$$

where A is an arbitrary analytic function. From this relation we can obtain u and v, as the solutions of the above system.

Remark. In paper [5], from which we have taken this example, KOLOSOV made an error. Stating that $f_j(\bar{z}) = P_j - iQ_j$, he, in fact, implicitely supposed that $\overline{f(z)} = f(\bar{z})$, which need not be true. However, if f is an analytic function, there is an analytic function g such that $\overline{f(z)} = g(\bar{z})$.

3) System

$$u_x - v_y + a(u, v)\, b(x, y) - c(u, v)\, d(x, y) = 0,$$

$$u_y + v_x + a(u, v)\, d(x, y) + c(u, v)\, b(x, y) = 0$$

where $a(u, v) + ic(u, v) = W(w)$, and $b(x, y) + id(x, y) = Z(z, \bar{z})$, can be written in the form

$$Dw = W(w)\, Z(z, \bar{z}),$$

and we can obtain the solution w, and therefore u and v, from the equality

$$\int \frac{dw}{W(w)} = \frac{1}{2} \int Z(z, \bar{z})\, d\bar{z} + A(z),$$

where A is an arbitrary analytic function.

It seems that the systems of partial differential equations which appear in the Plane Theory of Elasticity are in fact conjugate in the sense of KOLOSOV, since he has successfully used the above method in that theory. Though the expressions Dw, $\bar{D}w$ lie in the foundations of the theory of nonanalytic functions, KOLOSOV did not work on that theory, but used these expressions only to integrate systems of partial differential equations which he met in Elasticity.

KOLOSOV explained in detail his method of application of complex functions to the Theory of Elasticity in his book [6].

It is not difficult to show that

$$Dw = 2\frac{\partial w}{\partial \bar{z}}, \qquad \bar{D}w = 2\frac{\partial w}{\partial z}.$$

Therefore, conjugate equations of KOLOSOV, i.e. equations whose complex form is

$$F(z, \bar{z}, w, Dw) = 0$$

are in fact equations of the form

$$G\left(z, \bar{z}, w, \frac{\partial w}{\partial \bar{z}}\right) = F\left(z, \bar{z}, w, 2\frac{\partial w}{\partial \bar{z}}\right) = 0. \tag{6}$$

If we now treat z and $\bar{z}$ as indepenednt variables, equation (6) is a partial differential equation. It is therefore more appropriate to replace equation (6) by the following partial differential equation

$$G\left(x, y, u, \frac{\partial u}{\partial y}\right) = 0, \tag{7}$$

where $(x, y) \mapsto u(x, y)$ is the unknown function. However, since the only derivative contained in equation (6) (or (7)) is $\frac{\partial w}{\partial \bar{z}}$ $\left(\text{or } \frac{\partial u}{\partial y}\right)$, these equations are equivalent to an ordinary differential equation with a parameter.

This justifies KOLOSOV's method of integration.

It is interesting to note that though KOLOSOV was for a time unjustly forgotten, one can now find his name even in elementary secondary school text books, such as **[7]**.

3. A. BILIMOVIĆ of whom we spoke in **[1]** (references **[35]**—**[47]** in **[1]** belong to BILIMOVIĆ) was at the time of writing the mentioned papers apparently unaware of KOLOSOV's results. His latest paper on that subject **[8]** shows, however, that he has become acquainted with the work of KOLOSOV. (BILIMOVIĆ cites in **[8]** eight papers of KOLOSOV). It is therefore surprising to note that BILIMOVIĆ has not changed his former views on the history of nonanalytic functions, and that he makes no attempt in **[8]** to correct his assertion regarding the importance of the ,,Belgrade school''.

4. A completely different approach was adopted by F. TRICOMI. In his paper **[9]** he introduces the following expressions

$$\varepsilon = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \quad \eta = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \delta = 2\sqrt{\varepsilon^2 + \eta^2},$$

and calls δ *errore di analiticità*. He then proceeds to examine some properties of functions for which $\delta \neq 0$, i.e., nonanalytic functions. He is particularly interested in the case when δ is small.

The main result of TRICOMI's paper **[9]** is the following generalization of SCHWARZ' lemma (which we cite after TRICOMI):

Se la transformata del cerchio C è contenuta in un cerchio di centro O e raggio $1-h\,(h>0)$, e se di più in tutto C, l'errore di analiticità δ di w è minore di $h|z|$, allora la transformazione connessa a w è attrattiva in tutto C, tal quale si trattasse di una transformazione conforme.

In view of Tricomi's result (and others which hold for nonanalytic functions) one can suppose that many, if not all, theorems of the theory of analytic functions have their analogy in the theory of nonanalytic functions, where in the latter case one must always expect to meet δ, which appears in such a way that setting $\delta=0$ yields the known theorem for analytic functions.

It should be noted that in connection with nonanalytic functions F. Tricomi mentions only E. Kasner and E. R. Hedrick of whom we spoke in **[1]**.

5. An important contribution to the theory of nonanalytic functions was given recently by P. Caraman in the form of his monography **[10]**. It deals with nonanalytic functions from the modern aspect. Caraman introduces those functions by using geometrical reasonings — by means of n-dimensional quasiconformal mappings, which generalize conformal mappings realized by analytic functions. Book **[10]** also contains an interesting historical sketch (though mainly devoted to geometrical part of the theory) in which the attention is once again drawn to the often repeated error in connection with Beltrami's priority over Picard. Caraman's book contains a very extensive literature (over 1000 cited works). However, there is no mention of Kolosov or Tricomi. This suggests that there are more articles devoted to the subject of nonanalytic functions which are more or less forgotten. Notice that after the publication of his book **[10]**, P. Caraman published two more articles (**[11]**, **[12]**) on that subject.

6. Reference **[23]** in paper **[1]** is incorrect. It should be altered so that it reads:

[23] N. Théodorescu: *Derivée et primitives aréolaires.* Ann. Mat. Pur. Appl. Ser. 4, **49** (1960), 261—381.

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Dr P. Caraman has been kind enough to read this article in manuscript. Remarks and comments which he has given proved to be very useful in clarifying the text, and we thank Dr. Caraman for his assistance.

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