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307. ON GEODESIC LINES OF A CLASS OF SURFACES, II*

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Projections of geodesic lines of the surface

$$z=f(x, y) \tag{1}$$

onto the plane Oxy can be obtained by integration of the following differential equation

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & -1 \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix} = 0 \tag{2}$$

where dz and d^2z should be determined from (1).

Taking that $y=y(x)$, equation (2) becomes

$$(1+p^2+q^2)\left(\frac{d^2y}{dx^2}\right)+pt\left(\frac{dy}{dx}\right)^3+(qt-2ps)\left(\frac{dy}{dx}\right)^2+(2qs-pr)\left(\frac{dy}{dx}\right)+qr=0, \tag{3}$$

where p, q, r, s, t are classic notations for partial derivatives of (1).

In general case it is impossible to integrate (3).

In paper [1] it was shown that for surfaces for which $t=0$, i.e., for surfaces of the form

$$z=u(x)\,y+v(x),$$

where $u(x)=k=$ constant, equation (3) reduces to an integrable equation.

In this article we shall determine the conditions which have to be satisfied by u and v in order that among the projections of the geodesic lines of (1) onto Oxy plane there is one given by

$$y\,\frac{du(x)}{dx}+\frac{dv(x)}{dx}=0.$$

Equation (3) then becomes

$$(1+u^2)\left(\frac{d^2y}{dx^2}\right)+2u\left(\frac{du}{dx}\right)\left(\frac{dy}{dx}\right)+u\left(y\left(\frac{d^2u}{dx^2}\right)+\left(\frac{d^2v}{dx^2}\right)\right)=0,$$

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i.e.,

(4) $$-\left(\frac{d^2}{dx^2}\left(\frac{v'}{u'}\right)\right)(1+u^2)+2u\left(\frac{du}{dx}\right)\left(\frac{d}{dx}\left(\frac{v'}{u'}\right)\right)-u\left(\frac{v'}{u'}\right)\left(\frac{d^2u}{dx^2}\right)+u\left(\frac{d^2v}{dx^2}\right)=0$$

$$\left(v'=\frac{dv}{dx},\quad u'=\frac{du}{dx}\right).$$

Putting

(5) $$v=F(u)\qquad\left(\frac{dv}{dx}=\left(\frac{du}{dx}\right)\left(\frac{dF}{du}\right),\quad\frac{d^2v}{dx^2}=\left(\frac{d^2u}{dx^2}\right)\left(\frac{dF}{du}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{d^2F}{du^2}\right)\right)$$

equation (4) becomes

(6) $$(1+u^2)\left(\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2F}{du^2}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{d^3F}{du^3}\right)\right)+u\left(\frac{du}{dx}\right)^2\left(\frac{d^2F}{du^2}\right)=0.$$

This equation is identically satisfied if $\frac{d^2F}{du^2}\equiv 0$, i.e., if $F(u)=au+b$. In that case $v=au+b$, and so we have the following result:

Among the projections of geodesic lines onto the plane Oxy of surfaces of the form

$$z=u(x)\,y+au(x)+b\qquad(a,\ b=\text{const})$$

there is a straight line $y=-a$.

Suppose now that $\frac{d^2F}{du^2}\not\equiv 0$. Putting $p=\frac{du}{dx},\ \frac{d^2u}{dx^2}=p\frac{dp}{du}$, (6) becomes

$$p\left(\frac{dp}{du}\right)\left(\frac{d^2F}{du^2}\right)+p^2\frac{d^3F}{du^3}(1+u^2)+up^2\frac{d^2F}{du^2}=0$$

or

$$\frac{dp}{p}=-\left(\frac{u}{1+u^2}+\frac{F'''(u)}{F''(u)}\right)du,\quad\text{with}\quad F''=\frac{d^2F}{du^2},\quad F'''=\frac{d^3F}{du^3},$$

from where we get

$$\log p=-\frac{1}{2}\log(1+u^2)-\log F''(u)+\log C\qquad(C=\text{const}>0)$$

or

$$p=\frac{C}{F''(u)\sqrt{1+u^2}}.$$

From this equality we have

$$F''(u)\sqrt{1+u^2}\,du=C\,dx,$$

or

(7) $$x=w(u),$$

with

(8) $$w(u)=\frac{1}{C}\int F''(u)\sqrt{1+u^2}\,du+D$$

where D is a constant of integration.

Inversion of (7) yields

(9) $$u=w^{-1}(x)$$

and by the hypotheses for u and v, we have

(10) $$v=F\big(w^{-1}(x)\big).$$

Therefore, among the projections of geodesic lines onto the Oxy plane for the surfaces of the form

$$z = w^{-1}(x)\, y + F\big(w^{-1}(x)\big)$$

where w is determined by (8), there is one given by

$$u'(x)\, y + v'(x) = 0,$$

where u and v are determined by (9) and (10).

EXAMPLE 1. Amang the projection of geodesic lines onto the Oxy plane of the surface

$$z = \sqrt{x}\, y + \frac{1}{2}\sqrt{x}\,\sqrt{1+x} + \frac{1}{2}\log\left|\sqrt{x} + \sqrt{1+x}\right|$$

there is one given by

$$\frac{1}{2\sqrt{x}}\, y + \frac{1+x}{2\sqrt{x}\,\sqrt{1+x}} = 0, \text{ i.e., } \quad y = -\sqrt{1+x}.$$

In this case we have $F(x) = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x + \sqrt{1+x^2}\right|.$

EXAMPLE 2. Among the projection of geodesic lines onto the Oxy plane of the surface

$$z = (x^{2/3} - 1)^{1/2}\, y + (x^{2/3} - 1)^{3/2}$$

there is one given by

$$y = 3\,(1 - x^{2/3}).$$

REFERENCE

1. D. S. MITRINOVIĆ: *Sur les lignes géodésiques d'une classe des surface.* Publ. Math. Univ. Belgrade 3 (1934), 167—170.

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