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306. COMPLEMENTS OF THE HÖLDER INEQUALITY FOR FINITE SUMS*

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1. Introduction

Given sequences $\vec{a}=(a_1, \ldots, a_n)$ and $\vec{b}=(b_1, \ldots, b_n)$ with $a_i \geqq 0$, $b_i \geqq 0$ the HÖLDER inequality for finite sums may be written

$$
(1) \qquad H_{p,q}(\vec{a}, \vec{b}) = \frac{(\vec{a}, \vec{b})}{\|\vec{a}\|_p \|\vec{b}\|_q} = \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}} \leqq 1
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p, q \geqq 1$. If $p, q \leqq 1$ and $a_i>0$, $b_i>0$ then the inequality (1) is reversed ([1] page 24). A complement of the HÖLDER inequality gives a lower bound on $H_{p,q}(\vec{a}, \vec{b})$ for $p, q \geqq 1$ and an upper bound for $p, q \leqq 1$. In [2] various bounds of this type were given for integrals of functions defined on an interval $0 \leqq x \leqq a$. As noted there the methods used for integrals may also be applied to the case of finite sums. The method however does not apply directly as stated in [2] but, as pointed out by Professor D. S. MITRINOVIĆ, there are certain technical difficulties occurring in the finite sum case which do not appear in the integral case. The purpose of this note is to elaborate on the finite sum case and to examine some of these difficulties.

2. The General Method

We shall say $\vec{a}<\vec{b}$ or equivalently $\vec{b}>\vec{a}$ if

$$
(2) \qquad \sum_{j=1}^{i} a_j \leqq \sum_{j=1}^{i} b_j \quad (i=1, \ldots\ n-1) \text{ and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i .
$$

We shall need the following Theorem due to KARAMATA (see [3] page 30—32)

* Presented May 5, 1970 by D. S. MITRINOVIĆ.

Theorem (KARAMATA). *Given vectors* $\vec{a}$ *and* $\vec{b}$ *with*

(i) $a_1 \geqq \cdots \geqq a_n$ *and* $b_1 \geqq \cdots \geqq b_n$ *and*

(ii) $\vec{a} > \vec{b}$.

Then

$$\varphi(a_1) + \cdots + \varphi(a_n) \geqq \varphi(b_1) + \cdots + \varphi(b_n)$$

for any continuous convex function $\varphi(x)$.

Just as in the integral case [2] this Theorem implies

$$\|\vec{a}\|_p = \|\vec{a}^-\|_p \leqq \|\vec{a}_0\|_p \text{ if } p \geqq 1 \text{ and } \vec{a}_0 > \vec{a} \tag{3}$$

and

$$\|\vec{a}\|_p = \|\vec{a}^-\|_p \geqq \|\vec{a}_0\|_p \text{ if } p \leqq 1 \text{ and } \vec{c}_0 > \vec{a}, \tag{4}$$

where $\vec{a}^-$ denotes the rearrangement of $\vec{a}$ into decreasing order. Similarly we denote the rearrangement of $\vec{a}$ into increasing order by $\vec{a}^+$. Thus (see [1]),

$$(\vec{a}^-, \vec{b}^+) \leqq (\vec{a}, \vec{b}) \leqq (\vec{a}^+, \vec{b}^+). \tag{5}$$

We now give the basic result. This result was stated without proof in [2].

Theorem 1. *Let* $\vec{a}, \vec{b}$ *be vectors having nonnegative components with*

$$\|\vec{a}\|_p,\ \|\vec{b}\|_q > 0.$$

Then

(i) *If* $\vec{a}_0 = (a_{10}, \ldots, a_{n0})$, $\vec{b}_0 = (b_{10}, \ldots, b_{n0})$ *with* a_{i0} *increasing and* b_{i0} *decreasing and* $\vec{a}_0 < \vec{a}^+$, $\vec{b}_0 > \vec{b}^-$, *then*

$$H_{p,q}(\vec{a}, \vec{b}) \geqq H_{p,q}(\vec{a}_0, \vec{b}_0) \text{ for } p,\ q \geqq 1.$$

(ii) *If however* a_{i0} *and* b_{i0} *are increasing with* $\vec{a}^+ > \vec{a}_0$ *and* $\vec{b}^+ > \vec{b}_0$, *then*

$$H_{p,q}(\vec{a}, \vec{b}) \leqq H_{p,q}(\vec{a}_0, \vec{b}_0) \text{ for } p,\ q \leqq 1.$$

Proof: We give the proof of (ii) first. In view of (5) and (4) we may assume that the vectors $\vec{a}$ and $\vec{b}$ are increasing. Furthermore we need only show $(\vec{a}, \vec{b}) \leqq (\vec{a}_0, \vec{b}_0)$. This will follow from the two inequalities

$$(\vec{a}, \vec{b}) \leqq (\vec{a}, \vec{b}_0) \leqq (\vec{a}_0, \vec{b}_0). \tag{6}$$

To prove the first of these inequalities we consider the summation by parts formula

$$(\vec{a}, \vec{b})-(\vec{a}, \vec{b}_0)=\sum_{i=1}^{n}(a_ib_i-a_ib_{i0})$$

$$=a_n\sum_{i=1}^{n}(b_i-b_{i0})+\sum_{i=1}^{n-1}(a_{i+1}-a_i)\left[\sum_{j=1}^{i}b_j-\sum_{j=1}^{i}b_{j0}\right].$$

Since $\vec{b}^{+}=\vec{b}>\vec{b}_0$ the first term on the right is zero and the second is non-positive. This proves $(\vec{a}, \vec{b})\leqq(a, \vec{b}_0)$. The last inequality in (6) can be proved in much the same way to complete the proof of (ii). The proof of (i) is similar and will not be given.

3. Applications

In [2] it is stated that if $\vec{a}$ and $\vec{b}$ are concave, that is

$$a_i\geqq\frac{a_{i-1}+a_{i+1}}{2} \text{ and } b_i\geqq\frac{b_{i-1}+b_{i+1}}{2} \quad (i=2, \dots, n-1),$$

then

(7) $$\sum_{i=1}^{n}a_ib_i\geqq\frac{n-2}{2n-1}\left(\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2\right)^{1/2}.$$

Simple examples $(\vec{a}=(0, 1, 0))$ show that if $\vec{a}$ is concave then $\vec{a}^{+}$ need not be concave. This fact points up a gap in the proof of (7) which was indicated in [2]. We can however use Theorem 1 to prove

Theorem 2. *Let* $\vec{a}=(a_1, \dots, a_n)$ *and* $\vec{b}=(b_1, \dots, b_n)$ *be nonnegative vectors such that* $\vec{a}^{+}=(a_1^{+}, \dots, a_n^{+})$ *and* $\vec{b}^{-}=(b_1^{-}, \dots, b_n^{-})$ *are concave,*

$$a_i^{+}\geqq\frac{a_{i-1}^{+}+a_{i+1}^{+}}{2} \text{ and } b_i^{-}\geqq\frac{b_{i-1}^{-}+b_{i+1}^{-}}{2} \quad (i=2, \dots, n-1).$$

Then

$$\sum_{i=1}^{n}a_ib_i\geqq\left(\frac{n-2}{2n-1}\right)\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}.$$

The inequality is sharp and equality holds if $a_i=i-1$ *and* $b_i=n-i$.

Proof: Clearly we may assume that a_i are increasing, b_i are decreasing and both are concave. We may also assume that $\vec{a}, \vec{b}$ are normalized so that

$$\sum_{i=1}^{n}a_i^2=\sum_{i=1}^{n}b_i^2=\frac{n(n-1)}{2}.$$

This implies then that $\vec{a} < \vec{a}_0 = (0, 1, \ldots, n-1)$ and $(n-1, \ldots, 1, 0) = \vec{b}_0 > \vec{b}$. Theorem 1 then implies

$$H_{2,2}(\vec{a}, \vec{b}) \geqq H_{2,2}(\vec{a}_0, \vec{b}_0) = \frac{n-2}{2n-1}.$$

This completes the proof.

We conjecture that this theorem remains true if we replace the assumption that $\vec{a}^+$, $\vec{b}^+$ be concave with the assumption that $\vec{a}$, $\vec{b}$ be concave. It seems difficult to apply our methods to this problem.

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REFERENCES

1. HARDY, LITTLEWOOD and POLYA: *Inequalities*, 2nd ed. Cambridge 1964.
2. D. C. BARNES: *Some Complements of Hölder's Inequality*. J. Math. Anal. Appl. **26** (1969), 82—87.
3. E. F. BECKENBACH and R. BELLMAN: *Inequalities*. New York 1965.

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The question of priority of the Theorem ascribed to J. KARAMATA by D. C. BARNES, has been discussed in the book:

D. S. MITRINOVIĆ: *Analytic Inequalities* (Grundlehren der Mathematischen Wissenschaften, Bd. 165), Berlin-Heidelberg-New York 1970, p. 169.