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304. AN APPLICATION OF LAGRANGE'S FORMULA FOR THE INVERSION OF POWER SERIES*

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Applying a limiting process to the identity

$$\sum_{k=1}^{n} (-1)^k A_k(c,\ tb)\, A_{n-k}\big(a+bk-k,\ (1-t)\,b\big) = A_n\big(a-c,\ (1-t)\,b\big),$$

where

$$A_k(a,\ b) = \frac{a}{a+bk}\binom{a+bk}{k},$$

H. W. Gould [2] proved the following identity

$$(1)\qquad \sum_{k=0}^{n} (-1)^k B_k(c,\ tb)\, B_{n-k}\big(a+bk,\ (1-t)\,b\big) = B_n\big(a-c,\ (1-t)\,b\big),$$

where

$$B_k(a,\ b) = \frac{a}{a+bk}\,\frac{(a+bk)^k}{k!}.$$

The identity (1) is valid for any numbers a, b, c, t, and for $a=c$ and $t=1$ reduces to an orthogonality relation

$$\sum_{k=0}^{n} B_k(a,\ b)\, B_{n-k}(a+bk,\ 0) = \binom{0}{n}.$$

The purpose of this note is to derive (1) directly by use of Lagrange's formula for the inversion of power series. We state without proof Lagrange's formula [1].

If $w=f(z)=\dfrac{z}{\Phi(z)}=\sum\limits_{n=1}^{\infty} a_n z^n$ is an analytic function, regular in a neighborhood of the point $z=0$, and satisfying the condition $a_1=f'(0)\neq 0$, then the

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equation $f(z)=w$ has a unique solution, regular in a neighborhood of the point $w=0$ of the form

$$z=g(w)=\sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}(\Phi(z))^n\right]_{z=0}.$$

More generally, if $F(z)$ is an analytic function regular in a neighborhood of the point $z=0$, then there is a neighborhood of the point $w=0$ in which

$$F(z)=F(0)+\sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(F'(z)\{\Phi(z)\}^n\right)\right]_{z=0}. \tag{2}$$

With (2) given, the proof of (1) becomes simple. Chosing $\Phi(z)=e^{tbz}$ and $F(z)=e^{cz}$, we obtain

$$\begin{aligned} e^{cz} &= 1+\sum_{k=1}^{\infty}\frac{w^k}{k!}\left[\frac{d^{k-1}}{dz^{k-1}}(ce^{cz}e^{tbkz})\right]_{z=0} \\ &= 1+\sum_{k=1}^{\infty}\frac{w^k}{k!}c(c+tbk)^{k-1}=\sum_{k=0}^{\infty}w^kB_k(c,tb) \end{aligned} \tag{3}$$

where $w=ze^{-tbz}$.

Replacing in (3) z by $-z$, we obtain

$$e^{-cz}=(e^{-z})^c=\sum_{k=0}^{\infty}(-1)^kB_k(c,tb)z^ke^{tbkz}, \tag{4}$$

which upon the multiplication of both sides by e^{az} yields

$$\begin{aligned} e^{az-cz} &= \sum_{k=0}^{\infty}(-1)^kB_k(c,tb)z^ke^{az+tbkz} \\ &= \sum_{k=0}^{\infty}(-1)^kB_k(c,tb)\left(\frac{z}{e^{(b-tb)z}}\right)^ke^{az+bzk} \end{aligned} \tag{5}$$

Using (2) once more with $F(z)=e^{az+bzk}$ and $\Phi(z)=e^{(b-tb)z}$, we get from (5)

$$\begin{aligned} e^{az-cz} &= \sum_{k=0}^{\infty}(-1)^kB_k(c,tb)v^k\sum_{j=0}^{\infty}v^jB_j\big(a+bk,\ (1-t)b\big) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^kB_k(c,tb)B_j\big(a+bk,\ (1-t)b\big)v^{k+j}, \end{aligned} \tag{6}$$

where $v=\dfrac{z}{e^{(b-tb)z}}$.

Letting $k+j=v$, we find from (6)

$$e^{az-cz}=\sum_{n=0}^{\infty}v^n\sum_{k=0}^{\infty}(-1)^kB_k(c,tb)B_{n-k}\big(a+bk,\ (1-t)b\big). \tag{7}$$

On the other hand, the application of (2) with $F(z)=e^{az-cz}$ and $\Phi(z)=$ $=e^{(b-tb)z}$ yields

$$e^{az-cz}=\sum_{n=0}^{\infty} v^n B_n\big(a-c,\ (1-t)\,b\big). \tag{8}$$

Equating the coefficients of v^n in (7) and (8), we obtain

$$\sum_{k=0}^{n}(-1)^k B_k(c,\ tb)\,B_{n-k}\big(a+bk,\ (1-t)\,b\big)=B_n\big(a-c,\ (1-t)\,b\big).$$

REFERENCES

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2. H. W. GOULD: *A new convolution formula and some new orthogonal relations for inversion of series.* Duke Math. J. **29** (1962), 393—404.

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