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280. ON A PROPERTY OF SOME METHODS OF SUMMABILITY*

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1. Let $\{\mu_\tau\}$, where $\tau \in T$, be a family of nonnegative measures on a σ-algebra $\mathfrak{S}$ of subsets of an abstract set E. In [4], the notion of equisplittability of this family was introduced and investigated. Here, we shall consider topologically equisplittable families of measures, taking any $\tau_0 \notin T$ and supposing $T_0 = T \cup \{\tau_0\}$ to be a topological space.

The family $\{\mu_\tau\}$ will be called topologically equisplittable in T_0, if there exists $\eta > 0$ such that for any sequence of numbers $\varepsilon_k \downarrow 0$ satisfying the inequalities $\varepsilon_k \leqslant \eta$, $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$ for all k, there exist constants $M \geqslant \delta > 0$ and a sequence of pairwise disjoint sets $A_k \in \mathfrak{S}$ for which

$$(1) \qquad \delta \varepsilon_k \leqslant \overline{\lim_{\tau \to \tau_0}} \mu_\tau A_k \leqslant M \varepsilon_k .$$

Here, $\overline{\lim}$ is defined by

$$\overline{\lim_{\tau \to \tau_0}} f(\tau) = \inf_U \sup_{\tau \in U \setminus \{\tau_0\}} f(\tau),$$

where U runs over the set of all neighbourhoods of τ_0 in T_0, and $f(\tau)$ is an extended real-valued function on T. It is easily seen that if we take the coarsest topology in T_0, then $\overline{\lim_{\tau \to \tau_0}} f(\tau) = \sup_{\tau \in T} f(\tau)$, and topological equisplittability of $\{\mu_\tau\}$ in T_0 is equivalent to equisplittability of $\{\mu_\tau\}$ in the sense of [4].

2. Let $a(t, \tau) \geqslant 0$ be a function on $\langle t_0, \infty) \times \langle \tau^*, \infty)$, LEBESGUE measurable in the variable t for every $\tau \geqslant \tau^*$. As $\mathfrak{S}$ we take the σ-algebra of all LEBESGUE measurable subsets of $E = \langle t_0, \infty)$, and we define $T = \langle \tau^*, \infty)$. If

$$(2) \qquad \mu_\tau A = \int_A a(t, \tau)\, dt$$

for each $A \in \mathfrak{S}$, then $\{\mu_\tau\}$, $\tau \geqslant \tau^*$, is a family of measures in $\mathfrak{S}$. We put $\tau_0 = \infty$, and we define T_0 as the compactification of T by the point τ_0.

Let $\{a_n\}$, $n = 1, 2, \ldots$, be an increasing sequence of numbers such that $a_1 = t_0$ and $a_{n+1} - a_n \geqslant 2\eta$, for an $\eta > 0$ and $n = 1, 2, \ldots$. The sequence of intervals $I_n = \langle a_n, a_{n+1})$, $n = 1, 2, \ldots$, will be called a partition of $\langle t_0, \infty)$, of diameter not less than 2η.

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Theorem 1. *Let us suppose there exist an $\eta>0$, a partition $I_n=\langle a_n, a_{n+1})$ of $\langle t_0, \infty)$ of diameter not less than 2η, and numbers $M\geqslant\delta>0$ such that for any ε, $0<\varepsilon\leqslant\eta$, there exists ϑ, $0<\vartheta\leqslant\varepsilon$, satisfying the following condition: for each sequence of intervals $\langle\alpha_n, \beta_n)\subset I_n$, $\beta_n-\alpha_n=\vartheta(a_{n+1}-a_n)$, $n=1, 2, \ldots$, there holds*

$$\delta\varepsilon\leqslant \overline{\lim_{\tau\to\infty}} \int_A a(t,\tau)\,dt\leqslant M\varepsilon,$$

where $A=\bigcup_{n=1}^{\infty}\langle\alpha_n, \beta_n)$. Then the family of measures $\{\mu_\tau\}$ defined by (2) is topologically equisplittable in T_0.

Proof. Let a sequence $\varepsilon_k\downarrow 0$, $\varepsilon_k\leqslant\eta$, $\varepsilon_{k+1}/\varepsilon_k<\frac{1}{2}$ be given. We take $\varepsilon=\varepsilon_1$, and we choose $\alpha_n^1=a_n$, $\beta_n^1=(1-\vartheta_1)a_n+\vartheta_1 a_{n+1}$, where ϑ_1 corresponds to ε_1. By the assumption, the set $A_1=\bigcup_{n=1}^{\infty}\langle\alpha_n^1, \beta_n^1)$ satisfies the inequalities (1) for $k=1$. Now, let us suppose the sets $A_k=\bigcup_{n=1}^{\infty}\langle\alpha_n^k, \beta_n^k)$ are defined for $k=1, 2, \ldots, m-1$ in such a manner that $\beta_n^k=\alpha_n^{k+1}$, $\beta_n^k-\alpha_n^k=\vartheta_k(a_{n+1}-a_n)$, where $0<\vartheta_k\leqslant\varepsilon_k$ and that (1) holds. Since

$$\sum_{k=1}^{m-1}(\beta_n^k-\alpha_n^k)\leqslant(a_{n+1}-a_n)\sum_{k=1}^{m-1}\varepsilon_k<a_{n+1}-a_n,$$

we have $\langle\alpha_n^k, \beta_n^k)\subset I_n$ for $k=1, 2, \ldots, m-1$. We define $A_m=\bigcup_{n=1}^{\infty}\langle\alpha_n^m, \beta_n^m)$, where $\alpha_n^m=\beta_n^{m-1}$, $\beta_n^m=\beta_n^{m-1}+\vartheta_m(a_{n+1}-a_n)$. Then $\langle\alpha_n^m, \beta_n^m)\subset I_n$, and A_m satisfies the inequalities (1) with $k=m$.

The above theorem will be applied to prove the following theorems concerning concrete kernels connected with methods of summability of CESÀRO, STIELTJES and ABEL-LAPLACE (see [6], p. 134).

Theorem 2. *Let $a(t,\tau)=\frac{k}{\tau}\left(1-\frac{t}{\tau}\right)^{k-1}$ for $0\leqslant t\leqslant\tau$, $a(t,\tau)=0$ for $\tau<t$ where $k\geqslant 1$. Then the family of measures (2) is topologically equisplittable in T_0.*

Proof. We apply Theorem 1 with $I_n=\langle n-1, n)$, $\eta=\frac{1}{2}$, $\varepsilon\leqslant\frac{1}{2}$, $\vartheta=\frac{\varepsilon}{k}$, $\delta=4^{-k}$, $M=1$. We consider two cases: $\beta_{n-1}\leqslant\tau\leqslant\alpha_n$ and $\alpha_n<\tau<\beta_n$. In the first case we have

$$\int_A a(t,\tau)\,dt=\sum_{i=1}^{n-1}\left\{\left(1-\frac{\alpha_i}{\tau}\right)^k-\left(1-\frac{\beta_i}{\tau}\right)^k\right\}\leqslant\frac{\vartheta}{\tau}k(n-1)\leqslant\varepsilon\,\frac{\tau+1}{\tau}\to\varepsilon \text{ as } \tau\to\infty,$$

$$\int_A a(t,\tau)\,dt\geqslant\frac{\vartheta}{\tau}k\sum_{i=1}^{\left[\frac{1}{2}(n-1)\right]}\left(1-\frac{\gamma_i}{\tau}\right)^{k-1}\geqslant\frac{\varepsilon}{\tau}\left[\frac{1}{2}(n-1)\right]\cdot 4^{1-k}\geqslant\varepsilon\,\frac{\tau-3}{\tau}\cdot 4^{-k}\geqslant\delta\varepsilon,$$

where $\alpha_i \leqslant \gamma_i \leqslant \beta_i$ and $\delta = 4^{-k}$. In the second case we obtain in a similar way

$$\delta\varepsilon \leqslant \int_A a(t, \tau)\, dt \leqslant \varepsilon + \tau^{-k} \to \varepsilon \text{ as } \tau \to \infty.$$

Theorem 3. *If* $a(t, \tau) = \frac{\varrho}{\tau}\left(1 + \frac{t}{\tau}\right)^{-\varrho - 1}$, *where* $\varrho > 0$, *then the family of measures* (2) *is topologically equisplittable in* T_0.

Proof. We apply Theorem 1 with $I_n = \langle n-1, n)$, $\eta = \frac{1}{2}$, $\varepsilon \leqslant \frac{1}{2}$, $\vartheta = \varepsilon$, $\delta = M = 1$. We have

$$\int_A a(t, \tau)\, dt = \frac{\varrho\vartheta}{\tau} \sum_{n=1}^{\infty} \left(1 + \frac{\gamma_n}{\tau}\right)^{-\varrho - 1} \leqslant \frac{\varrho\varepsilon}{\tau}\left(1 + \frac{\tau}{\varrho}\right) \to \varepsilon \text{ as } \tau \to \infty,$$

where $\alpha_n \leqslant \gamma_n \leqslant \beta_n$, and

$$\int_A a(t, \tau)\, dt \geqslant \frac{\varrho\varepsilon}{\tau} \cdot \frac{\tau}{\varrho}\left(1 + \frac{1}{\tau}\right)^{-\varrho} \to \varepsilon \text{ as } \tau \to \infty.$$

Theorem 4. *If* $a(t, \tau) = \tau^{-1} e^{-t/\tau}$, *then the family of measures* (2) *is topologically equisplittable in* T_0.

Proof. With the same notation as in Theorem 3, we get

$$\varepsilon e^{-1/\tau} \leqslant \int_A a(t, \tau)\, dt \leqslant \varepsilon \frac{1 + \tau}{\tau},$$

which proves Theorem 4.

3. Let $(a_{n\nu})$, $n, \nu = 1, 2, \ldots$, be a nonnegative matrix, and let $\mathfrak{S}$ be the σ-algebra of all subsets of the set of positive integers E, $T = E$. If we take

(3) $$\mu_n A = \sum_i a_{n\nu_i} \text{ for } A = \{\nu_i\} \in \mathfrak{S},\ \mu_n \emptyset = 0,$$

then $\{\mu_n\}$, $n \in T$, is a family of measures in $\mathfrak{S}$. We put $\tau_0 = \infty$, and we define the topology in $T_0 = T \cup \{\tau_0\}$ by means of the filter of complements of finite subsets of T. As an example, we give sufficient conditions for topological equisplittability of such families in case of matrices corresponding to RIESZ methods of summability $(R, p, 1)$:

(4) $$a_{n\nu} = \begin{cases} \dfrac{p_\nu}{p_1 + \cdots + p_n} & \text{for } n \geqslant \nu, \\ 0 & \text{for } n < \nu \end{cases}$$

where $p_n > 0$, $P_n = p_1 + \cdots + p_n \to \infty$.

Theorem 5. *If* $\{p_n\}$ *is monotone and if there exist positive constants* a, b, α, β *such that*

$$a \leqslant \frac{np_n}{P_n} \leqslant b, \qquad \alpha \leqslant \frac{p^{2n+1}}{p_{2n}} \leqslant \beta \qquad (n = 1, 2, \ldots)$$

then the family $\{\mu_n\}$ *of measures* (3) *with* $a_{n\nu}$ *defined by* (4) *is topologically equisplittable in* T_0.

Proof. We limit ourselves to the case of increasing $\{p_n\}$. Let I_r be the set of integers n satisfying the inequalities $2^r \leqslant n < 2^{r+1}$, $r = 1, 2, \ldots$. and let $B_{kr} = \{j: 2^r + \cdots + 2^{r-k+1} \leqslant j \leqslant 2^r + \cdots 2^{r-k} - 1\}$ for $r \geqslant k$. $A_k = \bigcup_{r=k}^{\infty} B_{kr}$. Since $B_{kr} \subset I_r$ and B_{kr} are disjoint for $k = 1, 2, \ldots, r$, the sets $A_1, A_2, \ldots,$ are pairwise disjoint. Let $2^r < n \leqslant 2^{r+1}$, then

$$\sum_{\nu \in A_k} a_{n\nu} = \frac{1}{P_n} \sum_{i=k}^{r-1} \sum_{j \in B_{ki}} p_j + \varepsilon_{kn} \qquad \text{for } k \leqslant r-1,$$

where

$$0 \leqslant \varepsilon_{kn} \leqslant \frac{1}{P_n} \sum_{j \in B_{kr}} p_j.$$

Hence

$$\overline{\lim_{n\to\infty}} \sum_{\nu \in A_k} a_{n\nu} \leqslant \overline{\lim_{r\to\infty}} \frac{1}{P_{2^r}} \sum_{i=k}^{r-1} 2^{i-k} \cdot p_{2^{i+1} - 2^{i-k} - 1}$$

$$+ \overline{\lim_{r\to\infty}} \frac{1}{P_{2^r}} \cdot 2^{r-k} \cdot p_{2^{r+1} - 2^{r-k} - 1}$$

$$\leqslant \frac{b}{2^k} \overline{\lim_{r\to\infty}} \sum_{i=k}^{r-1} \frac{P_{2^{i+1} - 2^{i-k} - 1}}{P_{2^r}} + \frac{b}{2^k} \cdot \overline{\lim_{r\to\infty}} \frac{p_{2^{r+1}}}{p_{2^r}}$$

$$\leqslant \frac{b}{2^k} \overline{\lim_{r\to\infty}} \frac{P_{2^{r+1} - 2^{r-k} - 1}}{p_{2^r+1} + \cdots + p_{2^{r+1}}} + \frac{b\beta}{2^k} \leqslant \frac{b}{2^k} \cdot \frac{2\beta}{a} + \frac{b\beta}{2^k},$$

and

$$\overline{\lim_{n\to\infty}} \sum_{\nu \in A_k} a_{n\nu} \geqslant \overline{\lim_{r\to\infty}} \frac{1}{P_{2^{r+1}}} \sum_{i=k}^{r-1} 2^{i-k} \cdot p_{2^{i+1} - 2^{i-k+1} - 1}$$

$$\geqslant \frac{a}{2^{k+1}} \overline{\lim_{r\to\infty}} \sum_{i=k}^{r-1} \frac{P_{2^{i+1} - 2^{i-k+1} - 1}}{P_{2^{r+1}}}$$

$$\geqslant \frac{a}{2^{k+1}} \lim_{r\to\infty} \frac{P_{2^r - 1}}{p_{2^{r+1}+1} + \cdots + p_{2^{r+2}}} \geqslant \frac{a}{2^{k+1}} \cdot \frac{1}{4b\beta^3}.$$

Thus,

$$\delta\varepsilon_k \leqslant \overline{\lim_{n\to\infty}} \sum_{\nu \in A_k} a_{n\nu} \leqslant M\varepsilon_k,$$

where

$$\delta = \frac{a}{8b\beta^3}, \qquad M = b\beta\left(\frac{2}{a} + 1\right), \qquad \varepsilon_k = \frac{1}{2^k}.$$

Remark. Let us see that for instance the sequence $p_n = n^\gamma$, $\gamma \geqslant 0$, satisfies the assumptions of Theorem 1. If $\gamma = 0$, we get the $(C, 1)$-means.

4. In [4], the notion of equisplittability of a family of measures was applied to investigate some countably modulared spaces connected with strong summability.

If ϱ_i $(i=1, 2, \ldots)$ are pseudomodulars in a real linear space X (see [3], p. 49) such that $\varrho_i(\lambda x) = 0$ for $i = 1, 2, \ldots$ and all $\lambda > 0$ implies $x = 0$, then one defines $\varrho(x) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x)(1+\varrho_i(x))^{-1}$ and $\varrho_0(x) = \sup_i \varrho_i(x)$. X_ϱ and X_{ϱ_0} are modular spaces defined by means of modulars ϱ and ϱ_0, respectively; X_ϱ is called a countably modulared space, and X_{ϱ_0} — a uniformly countably modulared space (see [2]).

Let $\mathfrak{S}$ be a σ-algebra of subsets of an abstract set E. and let X be the space of real functions x defined on E, measurable with respect to $\mathfrak{S}$. In [4] we considered pseudomodulars $\varrho_i(x) = \sup_\tau \int_E \varphi_i(|x(t)|) d\mu_\tau$, where $\{\mu_\tau\}$, $\tau \in T$, is a family of uniformly bounded measures on $\mathfrak{S}$ and $\varphi_i(u)$ are φ-functions.

Here, we shall suppose $T_0 = T \cup \{\tau_0\}$, where $\tau_0 \notin T$, to be a topological space and we shall define

$$\varrho_i(x) = \overline{\lim_{\tau \to \tau_0}} \int_E \varphi_i(|x(t)|)\, d\mu_\tau. \tag{5}$$

According to the definition of $\overline{\lim}$, we have

$$\varrho_i(x) = \inf_U \sup_{\tau \in U \setminus \{\tau_0\}} \int_E \varphi_i(|x(t)|)\, d\mu_\tau,$$

where U runs over all neighborhoods of the point τ_0. It is easily seen that if we take the coarsest topology in T_0, then the pseudomodulars ϱ_i are reduced to those considered in [4].

Now, let X_ϱ and X_{ϱ_0} be the countably modulared spaces and the uniformly countably modulared space defined by the sequence of modulars (5). The following necessary and sufficient conditions for identity $X_\varrho = X_{\varrho_0}$ may be proved in the same manner as in [4]:

Theorem 6. *Let family of measures $\{\mu_\tau\}$ be uniformly bounded and let the φ-functions φ_i be equicontinuous at* 0 *and satisfy the condition*:

(*) *there are constants k, c, $u_0 > 0$ and an index i_0 such that $\varphi_i(cu) \leqslant k\varphi_{i_0}(u)$ for $u \geqslant u_0$ and $i \geqslant i_0$. Then $X_\varrho = X_{\varrho_0}$.*

Theorem 7. *Let the family of measures $\{\mu_\tau\}$ be uniformly bounded and topologically equisplittable in T_0, and let the φ-functions φ_i satisfy the following conditions*:

1) *for every index i there are constants λ_i, β_i, $\vartheta_i > 0$ such that $\varphi_i(\lambda_i u) \leqslant \beta_i \varphi_k(u)$ for every $u \geqslant \vartheta_i$ and $k \geqslant i$,*

2) *for every $\varepsilon > 0$ there are numbers u_ε, $a_\varepsilon > 0$ depending on i such that $\varphi_i(au) < \varepsilon\varphi_i(u)$ for $0 \leqslant a \leqslant a_\varepsilon$, $u \geqslant u_\varepsilon$. Then $X_\varrho = X_{\varrho_0}$ implies* (*).

Let us remark that in case T_0 defined as in 2, Theorem 2, $k = 1$, we obtain pseudomodulars corresponding to the MARCINKIEWICZ-ORLICZ space (see [1], p. 12; [3], p. 63, and [5], p. 188).

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