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278. REMARKS ON THE MONODROMY THEOREM*

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Studies in analytic continuation lead one, more or less naturally, to the monodromy theorem. The reader will recall that if two arcs with the same end points are homotopic in a region Ω, then analytic continuation along either arc of a function element (f, Ω_0) always leads to the same result (germ) provided that analytic continuation is possible along all intermediate arcs. A function element (f, Ω_0) is simply an analytic function f in the region $\Omega_0 \subset \Omega$. If Ω is simply connected and analytic continuation of some function element (f, Ω_0) is possible along all arcs issuing from a point $z_0 \in \Omega_0$, then the monodromy theorem asserts that these continuations define an analytic function in Ω which agrees with f in Ω_0.

The particular value of this theorem — and principal source of its abuse — occurs in the case where one is attempting to invert an analytic function whose derivative is known to be nonvanishing. A particular example of this occurs in [**1**, § 285] where it is shown that a finite BLASCHKE product

$$B(z) = \prod_{k=1}^{n} \left[\frac{a_k - z}{1 - \bar{a}_k z} \right]$$

$|a_k| < 1$, $k = 1, \ldots, n$, must satisfy $B'(z) = 0$ somewhere in $|z| < 1$ or else $n = 1$. The assumption that B' never vanishes allows one to select local inverses at each point of the unit disc. A standard argument may be used to show that analytic continuation along all arcs is then possible. Since B maps the disc onto the disc, the monodromy theorem shows that B must be invertible.

In view of this one may ask to what extent does this generalize? Suppose, for example, that f maps the unit disc conformally (that is, f' never vanishes) onto itself. Is f univalent? The answer, undoubtedly known for generations, is no. However, it would seem to the perspicuous reader that the proof used above for B should also carry over to the following situation: if f maps the unit disc conformally onto itself, and if $f^{-1}(w)$ is a finite set for each $|w| < 1$, then f is univalent. This, too, is false. In fact the cardinality of $f^{-1}(w)$ can be bounded and still f need not be univalent. The only positive assertion one can make is this: If $f^{-1}(w)$ has the *same* (finite) cardinality for each w, and f maps a simply connected region conformally onto a simply connected region, then f is univalent.

In order to exhibit an elementary conformal mapping which is not univalent it will be convenient to work regions other than the unit disc and appeal to the RIEMANN mapping theorem at the conclusion.

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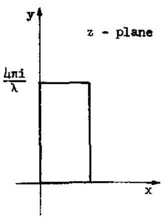
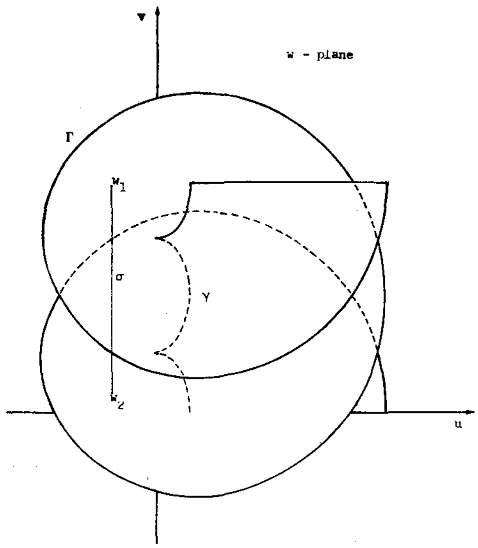
Choose a real number $\lambda \geqslant 2.25$ and set $w(z) = \exp[\lambda z] + \lambda z + 1$. Let $z = x + iy$ be confined to the rectangle $0 < x < 1$, $0 < y < \frac{4\pi}{\lambda}$. The image of this rectangle is sketched in the accompanying figure. The inner boundary arc γ is easily sketched since its equation is simply $w(iy) = (1 + \cos \lambda y) + i(\lambda y + \sin \lambda y)$. Any calculus student will recognize this to be a cycloid.





The „outer" boundary Γ, given by $w(1 + iy) = \lambda + (1 + e^{\lambda} \cos \lambda y) + i(\lambda y + e^{\lambda} \sin \lambda y)$ is a prolate cycloid traced out by a point moving on the flange of a wheel of radius 1 at a distance e^{λ} from the center. The wheel, however, rolls along the line $u = \lambda$, not $u = 0$.

It might be emphasized that the image of the smaller square $0 < x < 1$, $0 < y < \frac{2\pi}{\lambda}$ is not simply connected. The point πi is omitted. In order to compensate for this, one increases the domain until a second revolution is traversed.

While it is true that another cusp is introduced at $2\pi i$, this point was already covered by the first revolution. The point πi is now seen to be covered as well and the resulting image is simply connected.

Finally, note that $w'(z) = \lambda \exp[\lambda z] + \lambda$ does not vanish in the given rectangle.

The interested reader may find it both amusing and instructive to attempt analytic continuation of an inverse to f along the arc σ from w_1 to w_2. A careful reading of § 233 in [**1**] is now recommended.

In conclusion it might also be remarked that a lower bound on λ is not easily calculated. Elementary estimates show that $\lambda > 2$ is necessary in that when $\lambda = 2$, the corresponding range fails to be simply connected.

REFERENCE

[1] C. Carathéodory, *Theory of Functions of a Complex Variable* (translated by F. Steinhardt), Vols. I, II, New York 1954.