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277. IMPROVEMENTS OF STIRLING'S FORMULA BY ELEMENTARY METHODS*

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Since STIRLING'S time (1764) it has been known that $n!\sim C\sqrt{n}\,(n/e)^n$, with $C=\sqrt{2\pi}$, that is

$$\lim \frac{n!}{\sqrt{2\pi n}\,(n/e)^n}=1. \tag{1}$$

Formula (1) is known as *Stirling's formula*, and a great many elementary proofs of it have been given. Most such proofs use WALLIS' formula

$$\lim \frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdots (2n-1)(2n+1)}=\frac{\pi}{2} \tag{2}$$

to show that $C=\sqrt{2\pi}$. However, one of the most recent elementary proofs, by W. FELLER [**2**, **3**], avoids any appeal to WALLIS' formula. Since completely elementary proofs of (2) are well known (see, for example [**9**]) this does not seem to be essential.

In addition, a number of upper and lower bounds for $n!$ (most of which imply (1)) have been obtained by various authors, also by elementary methods. One of the most elementary of these is the estimate $e^{11/12}\sqrt{n}\,(n/e)^n<n!<e\sqrt{n}\,(n/e)^n$ obtained by HUMMEL [**4**] in 1940. Most bounds are of the form

$$\sqrt{2\pi n}\,(n/e)^n e^{\alpha_n}<n!<\sqrt{2\pi n}\,(n/e)^n e^{\beta_n}, \tag{3}$$

where α_n and β_n tend to zero through positive values. For example, $\beta_n=(12n)^{-1}$ was proved in each of [**1**, **5**, **7**, **8**, **10**], while the successively better values $\alpha_n=1/(12n+6)$, $1/(12n+1)$, $1\Big/\left(12n+\frac{1}{4}\right)$, $1\Big/\left(12n+\frac{3}{2(2n+1)}\right)$ and $1/(12n)-1/(360n^3)$ were obtained in [**10**], [**8**], [**1**], [**5**] and [**7**] respectively. The method of proof is essentially the same in all of these cases and appears to be due to CESÀRO [**1**]; in particular, WALLIS' formula is used. (Cf. also MITRINOVIĆ [**6**] where a more extensive bibliography is given.) In this note we give a further refinement of CESÀRO's method to prove (3) with

$$\alpha_n=\frac{1}{12n}-\frac{1}{360n^3},\quad \beta_n=\frac{1}{12n}-\frac{1}{(360+\gamma_n)n^3},\quad \gamma_n=30\,\frac{7n(n+1)+1}{n^2(n+1)^2} \tag{4}$$

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for $n \geqq 1$. We thus obtain the best lower bound (for $n \geqq 2$) of those noted above, and a substantially improved upper bound.

In view of (1) we begin with the sequence $\{a_n\}$ defined by

$$a_n = \frac{n!\, e^n}{n^n \sqrt{n}} \qquad n \geqq 1,$$

for which

$$\frac{a_n}{a_{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \to 1. \tag{5}$$

We shall prove — in accordance with (1) — that $\lim a_n$ exists and has the value $\sqrt{2\pi}$, at the same time obtaining the bounds (3), (4). To this end, we shall obtain bounds for

$$\log\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right).$$

From the well-known expansion

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}, \quad |x|<1,$$

we obtain on setting $x=(2n+1)^{-1}$,

$$\log\left(\frac{n+1}{n}\right) = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\,\frac{1}{(2n+1)^{2k-1}} = \frac{1}{n+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\,\frac{1}{(2n+1)^{2k}},$$

or

$$\log\left\{\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\right\} = \sum_{k=1}^{\infty}\frac{1}{2k+1}\,\frac{1}{(2n+1)^{2k}}. \tag{6}$$

Note that (5) and (6) show that $a_n > a_{n+1}$ for all $n \geqq 1$. The idea of the proof is now to find positive sequences $\{f(n)\}$, $\{g(n)\}$ both of which tend to zero, such that

$$f(n)-f(n+1) < \sum_{k=1}^{\infty}\frac{1}{2k+1}\,\frac{1}{(2n+1)^{2k}} < g(n)-g(n+1) \tag{7}$$

holds at least for all sufficiently large n, say $n \geqq N$. For then it follows from (5) and (6) that

$$\exp\{f(n)-f(n+1)\} < \frac{a_n}{a_{n+1}} < \exp\{g(n)-g(n+1)\},$$

and hence both

$$a_{n+1}\exp\{-f(n+1)\} < a_n \exp\{-f(n)\} \equiv x_n, \tag{8}$$

$$a_{n+1}\exp\{-g(n+1)\} > a_n \exp\{-g(n)\} \equiv y_n \tag{9}$$

for $n \geqslant N$. The sequence $\{x_n\}$ is thus monotone decreasing and bounded below by zero so that $\lim x_n = \alpha$ exists with $\alpha \geqslant 0$. Since $f(n) \to 0+$, it also follows that

$$\lim a_n = \lim x_n \exp\{f(n)\} = \alpha \cdot 1 = \alpha$$

also exists. Similarly, by (9), the sequence $\{y_n\}$ is monotone increasing with $y_{n+1} < a_{n+1} < a_n < \cdots < a_1$, so that $\lim y_n = \beta$ exists with $\beta > 0$. Hence,

$$\alpha = \lim a_n = \lim y_n \exp\{g(n)\} = \beta > 0.$$

Using WALLIS' formula (2), and the fact that $n! = (n/e)^n \sqrt{n}\, a_n$, we have

$$\pi = 2 \lim \frac{\{2^{2n}(n!)^2\}^2}{\{(2n)!\}^2(2n+1)} = \lim \frac{1}{1+\dfrac{1}{2n}} \frac{\{2^{2n}(n!)^2\}^2}{\{(2n)!\}^2 n},$$

whence

$$\sqrt{\pi} = \lim \frac{2^{2n}(n!)^2}{(2n!)\sqrt{n}} = \lim \frac{2^{2n}(n/e)^{2n} n a_n^2}{(2n/e)^{2n}\sqrt{2}\, n a_{2n}} = \lim \frac{a_n^2}{\sqrt{2}\, a_{2n}} = \frac{\alpha}{\sqrt{2}},$$

since $\alpha > 0$. Hence, $\lim a_n = \lim x_n = \lim y_n = \alpha = \sqrt{2\pi}$ so, using the monotone character of $\{x_n\}$ and $\{y_n\}$, it follows that

$$y_n = a_n \exp\{-g(n)\} < \alpha = \sqrt{2\pi} < a_n \exp\{-f(n)\} = x_n,$$

$$\frac{n!\, e^n}{n^n \sqrt{n}} \exp\{-g(n)\} < \sqrt{2\pi} < \frac{n!\, e^n}{n^n \sqrt{n}} \exp\{-f(n)\},$$

so that

$$\sqrt{2\pi n}\,(n/e)^n \exp\{f(n)\} < n! < \sqrt{2\pi n}\,(n/e)^n \exp\{g(n)\} \tag{10}$$

for all $n \geqslant N$, provided f and g satisfy (7) as specified.

Turning now to (7), we proceed as in CESÀRO [1], but carry one extra term in each case to obtain

$$\sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{1}{(2n+1)^{2k}} < \frac{1}{3(2n+1)^2} + \frac{1}{5} \sum_{k=2}^{\infty} \frac{1}{(2n+1)^{2k}}$$

$$= \frac{1}{3(2n+1)^2} + \frac{1}{5} \frac{(2n+1)^{-4}}{1-(2n+1)^{-2}}$$

$$= \frac{1}{3(2n+1)^2} + \frac{1}{20n(n+1)(2n+1)^2},$$

and similarly,

$$\sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{1}{(2n+1)^{2k}} > \frac{1}{3(2n+1)^2} + \frac{1}{5} \sum_{k=2}^{\infty} \left(\frac{5}{7}\right)^{k-2} \frac{1}{(2n+1)^{2k}}$$

$$= \frac{1}{3(2n+1)^2} + \frac{1}{5} \frac{(2n+1)^{-4}}{1-\dfrac{5}{7}(2n+1)^{-2}}$$

$$= \frac{1}{3(2n+1)^2} + \frac{7}{10(2n+1)^2(14n^2+14n+1)}$$

since $\frac{1}{5}\left(\frac{5}{7}\right)^{k-2}<\frac{1}{2k+1}$ for all $k>3$ is easily verified. In order to obtain (7), and with it the desired estimates (3) and (4), we now show that

$$\frac{1}{3(2n+1)^2}+\frac{7}{10(2n+1)^2(14n^2+14n+1)} \geqslant \frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{A}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right), \quad n\geqslant 1, \tag{11}$$

$$\frac{1}{3(2n+1)^2}+\frac{1}{20n(n+1)(2n+1)^2} \leqslant \frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{B}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right), \quad n\geqslant N\geqslant 1, \tag{12}$$

both hold, provided $0<A\leqslant 360$ and $B\geqslant 360+\gamma_N$. It will then follow that (7) holds with $f(n)=\frac{1}{12n}-\frac{1}{360n^3}=a_n$, and $g(n)=\frac{1}{12n}-\frac{1}{(360+\gamma_N)n^3}$. Hence (10) will also hold for such f and g (and $n\geqslant N$). But then, on setting $n=N\geqslant 1$ in (10), we obtain the estimates (3), (4) and the proof will be complete.

Now, (11) holds if and only if

$$\frac{1}{A}\,\frac{3n^2+3n+1}{n^3(n+1)^3}\geqslant\frac{1}{12n(n+1)}-\frac{1}{3(2n+1)^2}-\frac{7}{10(2n+1)^2(14n^2+14n+1)}$$

$$=\frac{28n^2+28n+5}{60n(n+1)(4n^2+4n+1)(14n^2+14n+1)}$$

or

$$A\leqslant\frac{60(3n^2+3n+1)(4n^2+4n+1)(14n^2+14n+1)}{n^2(n+1)^2(28n^2+28n+5)}.$$

It is easy to verify that the quotient on the right exceeds 360 for all $n\geqslant 1$ (set $y=n(n+1)$ to simplify!), so that (11) holds as asserted if $0<A\leqslant 360$. Similarly, (12) is equivalent to

$$\frac{1}{B}\,\frac{3n^2+3n+1}{n^2(n+1)^3}\leqslant\frac{1}{30n(n+1)(2n+1)^2}$$

or

$$B\geqslant\frac{30(3n^2+3n+1)(4n^2+4n+1)}{n^2(n+1)^2}=30\,\frac{12n^4+24n^3+19n^2+7n+1}{n^2(n+1)^2}$$

$$=30\,\frac{12n^2(n+1)^2+7n^2+7n+1}{n^2(n+1)^2}=360+30\,\frac{7n(n+1)+1}{n^2(n+1)^2}=360+\gamma_n.$$

For any $B>360$ this inequality will be satisfied for all sufficiently large n. Since $\gamma_n=30\{7[n(n+1)]^{-1}+[n(n+1)]^{-2}\}$ is a decreasing function of n, we have

$360+\gamma_N \geqslant 360+\gamma_n$ for all $n \geqslant N \geqslant 1$, so that (12) holds as asserted provided $B \geqslant 360+\gamma_N$.

By retaining additional terms in the expansion (6), the estimates (3), (4), can, of course, be improved. For example by retaining one more term — and with considerably more algebra — one obtains (3) with

$$\alpha_n = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260\left(1+\dfrac{2}{n(n+1)}\right)n^5}, \quad \beta_n = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}.$$

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