PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 274 — № 301 (1969)

276. INEQUALITIES INVOLVING ITERATED KERNELS AND CONVOLUTIONS*

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In 1960, ATKINSON, WATTERSON and MORAN [**1**] conjectured that if $K(x, y)$ is a nonnegative, symmetric kernel which is integrable on a square $[0, a] \times [0, a]$, and $K_n(x, y)$ is the n^{th} iterate of K, then

$$a^{n-1} \int_0^a \int_0^a K_n(x, y)\, dx\, dy \geqslant \Big(\int_0^a \int_0^a K(x, y)\, dx\, dy\Big)^n \text{ for } n = 1, 2, \ldots \tag{1}$$

This conjecture was proved for $n = 2$ and $n = 3$ by using appropriate matrix inequalities and applying these inequalities to approximating RIEMANN sums for the integrals appearing in (1); the truth of the conjecture for all n of the form $n = 2^r 3^s$ was then easily established by induction.

In this note we shall prove a somewhat weaker inequality than (1) which is valid for all $n \geqslant 1$ and for an arbitrary nonnegative kernel K. In addition, we use the same technique to obtain a lower bound for the convolution of n positive functions. However, before proceeding with this we want to point out that (1) may be false for all $n > 1$ if K is *not* symmetric. To see this, let $K(x, y) = f(x)\, g(y)$ where f, g are positive and continuous on $[0, a]$. Using the definition $K_n(x, y) = \int_0^a K_{n-1}(x, s)\, K(s, y)\, ds$ for $n \geqslant 2$, we obtain $K_n(x, y) = A^{n-1} f(x)\, g(y)$ where $A = \int_0^a f(s)\, g(s)\, ds$. Then for all $n \geqslant 2$, (1) is satisfied if, and only if,

$$a \int_0^a f(s)\, g(s)\, ds \geqslant \Big(\int_0^a f(s)\, ds\Big)\Big(\int_0^a g(s)\, ds\Big). \tag{2}$$

However, by CHEBYCHEV's inequality [**2**, Theorem 236], (2) does not hold if f and g are oppositely ordered, for example if $f(x) \equiv x$, $g(x) \equiv a - x$.

We shall formulate our inequality in a more general manner than that given in (1). To this end, let μ be a nonnegative measure on an σ-algebra of subsets of a set A, with $0 < \mu(A) < \infty$. For brevity, we write $A_2 = A \times A$,

* Presented September 15, 1969 by D. S. MITRINOVIĆ.

$A_3 = A \times A \times A, \ldots$, and for clarity we also write $dx_1\, dx_2 \cdots dx_n$, or $dx_0\, dx_1 \cdots dx_{n-1}$ etc. to denote integration with respect to the product measure $\mu \times \mu \times \cdots \times \mu$ defined on subsets of A_n.

Theorem 1. *Suppose that $K \in L_2(A_2)$ with $K(x, y) \geqslant 0$ on A_2. If the iterated kernels of K are defined by*

$$K_1(x, y) = K(x, y), \quad K_n(x, y) = \int_A K_{n-1}(x, r)\, K(r, y)\, dr \quad (n \geqslant 2),$$

then

$$\int_{A_2} K_n(x, y)\, dx\, dy \geqslant [\mu(A)]^{n+1} \exp\left\{n \int_{A_2} \log K(x, y)\, dx\, dy / \mu^2(A)\right\}. \tag{3}$$

Proof. It follows from the definition and FUBINI'S Theorem that

$$K_n(x_0, x_n) = \int_{A_{n-1}} \left(\prod_{j=1}^{n} K(x_{j-1}, x_j)\right) dx_{n-1} \cdots dx_1,$$

whence

$$\int_{A_2} K_n(x, y)\, dx\, dy = \int_{A_{n+1}} \left(\prod_{j=1}^{n} K(x_{j-1}, x_j)\right) dx_n \cdots dx_0. \tag{4}$$

We note that each K_n is defined a.e. on A_2 and is in $L_2(A_2)$ since $K \in L_2(A_2)$; in addition $K_n \in L(A_2)$ since $\mu(A) < \infty$, so the existence of the integral in (4) is assured. Now,

$$\lim_{r \to 0+} \left\{\frac{1}{n} \sum_{j=1}^{n} K^r(x_{j-1}, x_j)\right\}^{1/r} = \prod_{j=1}^{n} K^{\frac{1}{n}}(x_{j-1}, x_j) = \left\{\prod_{j=1}^{n} K(x_{j-1}, x_j)\right\}^{1/n},$$

by [2, Theorem 3], and the convergence is monotonic, so by (4) we obtain

$$\begin{aligned} \int_{A_2} K_n(x, y)\, dx\, dy &= \lim_{r \to 0+} \int_{A_{n+1}} \left\{\frac{1}{n} \sum_{j=1}^{n} K^r(x_{j-1}, x_j)\right\}^{\frac{n}{r}} dx_n \cdots dx_0 \\ &= \lim_{r \to 0+} X_r, \text{ say.} \end{aligned} \tag{5}$$

For $0 < r < n$ the function $x^{n/r}$ is convex, hence by JENSEN'S inequality [2, Theorem 204], with weight function $p \equiv 1$, we have

$$X_r \geqslant [\mu(A)]^{n+1} \left\{\int_{A_{n+1}} \frac{1}{n} \sum_{j=1}^{n} K^r(x_{j-1}, x_j)\, dx_n \cdots dx_0 / [\mu(A)]^{n+1}\right\}^{\frac{n}{r}}.$$

However,

$$\int_{A_{n+1}} K^r(x_{j-1}, x_j)\, dx_n \cdots dx_0 = [\mu(A)]^{n-1} \int_{A_2} K^r(x, y)\, dx\, dy, \tag{6}$$

so that

$$X_r \geqslant [\mu(A)]^{n+1-\frac{2n}{r}} \left(\int_{A_2} K^r(x, y)\, dx\, dy\right)^{\frac{n}{r}}.$$

Finally, we use [2, Theorem 184] with $p=1$ which gives

$$\left(\frac{\int\limits_{A_2} K^r(x,y)\,dx\,dy}{\int\limits_{A_2} dx\,dy}\right)^{\frac{1}{r}} \geqslant \exp\left(\frac{\int\limits_{A_2} \log K\,dx\,dy}{\int\limits_{A_2} dx\,dy}\right).$$

Combining this with the preceding inequality gives, for $0<r<n$,

$$X_r > [\mu(A)]^{n+1} \exp\left\{n \int\limits_{A_2} \log K\ dx\,dy/\mu^2(A)\right\}$$

so that (3) now follows from (5), and the proof is complete.

The method of proof used above is essentially that used by J. F. C. KINGMAN in [3]. A similar proof gives the inequality

$$(7)\qquad \int\limits_a^b \int\limits_a^x K_n(x,y)\,dy\,dx$$

$$\geqslant \frac{(b-a)^{n+1}}{(n+1)!} \exp\left\{\frac{n(n+1)}{(b-a)^{n+1}} \int\limits_a^b \int\limits_a^x (b+y-a-x)^{n-1} \log K(x,y)\,dy\,dx\right\}$$

which is val'd for all $n \geqslant 3$, when K is a square integrable, nonnegative VOLTERRA kernel on $[a,b]\times[a,b]$, so that $K(x,y)=0$ for $y>x$. The details of the proof are rather messy, and we only note the following facts as a help to the reader. First, the n^{th} iterated kernel of a VOLTERRA kernel is given by

$$K_n(t_n, t_0) = \int\limits_{t_0}^{t_n} \int\limits_{t_0}^{t_{n-1}} \cdots \int\limits_{t_0}^{t_2} K(t_n, t_{n-1}) \cdots K(t_1, t_0)\,dt_1 \cdots dt_{n-1}\quad (n \geqslant 2).$$

The essential step in the proof — corresponding to equation (6)—is now

$$\frac{1}{n} \int\limits_a^b \int\limits_{t_0}^b \int\limits_{t_0}^{t_n} \cdots \int\limits_{t_0}^{t_2} \sum_{j=1}^n K^r(t_j, t_{j-1})\,dt_1 \cdots dt_n\,dt_0$$

$$= \frac{1}{n!} \int\limits_a^b \int\limits_a^x (b+y-a-x)^{n-1} K^r(x,y)\,dy\,dx,$$

and this may be verified for all $n \geqslant 2$ by induction. We also note that the same technique may be applied to give a lower bound for K_n itself, either in Theorem 1, or for VOLTERRA kernels. The following theorem is an example of an inequality of this type.

Theorem 2. *Let $f_1, f_2, \dots, f_n$ be nonnegative functions each of which is Lebesgue square integrable on* $[0, a]$ *for all* $a>0$. *If the convolution* $\int_0^x f_i(t) f_j(x-t)\,dt$ *is denoted by* $f_i * f_j(x)$, *then for all* $n \geqq 2$, *and* $x \geqq 0$, *we have*

$$(8)\quad f_1 * f_2 * \cdots * f_n(x) \geqq \frac{x^{n-1}}{(n-1)!} \exp\left\{(n-1)\, x^{-n+1} \int_0^x (x-u)^{n-2} \sum_{j=1}^n \log f_j(u)\, du\right\}$$

Proof. Proceeding as in Theorem 1, we have (with $t_0 = 0$),

$$f_1 * f_2 * \cdots * f_n(t_n)$$
$$= \int_0^{t_n}\int_0^{t_{n-1}} \cdots \int_0^{t_2} f_1(t_1) f_2(t_2-t_1) \cdots f_n(t_n-t_{n-1})\, dt_1 \cdots dt_{n-1}$$
$$= \lim_{r\to 0+} \int_0^{t_n}\int_0^{t_{n-1}} \cdots \int_0^{t_2} \left\{\frac{1}{n}\sum_{j=1}^n f_j^r(t_j-t_{j-1})\right\}^{\frac{n}{r}} dt_1 \cdots dt_{n-1}$$
$$\geqq I_n(t_n) \lim_{r\to 0+} \left\{\int_0^{t_n} \cdots \int_0^{t_2} \sum_{j=1}^n f_j^r(t_j-t_{j-1})\, dt_1 \cdots dt_{n-1}/nI_n(t_n)\right\}^{\frac{n}{r}},$$

where $I_n(t_n) = \int_0^{t_n} \cdots \int_0^{t_2} dt_1 \cdots dt_{n-1} = t_n^{n-1}/(n-1)!$ We now note that

$$(9)\quad \int_0^{t_n} \cdots \int_0^{t_2} \sum_{j=1}^n f_j^r(t_j-t_{j-1})\, dt_1 \cdots dt_{n-1} = \int_0^{t_n} \frac{(t_n-u)^{n-2}}{(n-2)!} \sum_{j=1}^n f_j^r(u)\, du.$$

This is immediate for $n = 2$, and follows by an easy induction for all $n \geqq 2$. Hence we have

$$(10)\quad f_1 * f_2 * \cdots * f_n(x) \geqq \lim_{r\to 0+} [I_n(x)]^{1-\frac{n}{r}} \left\{\frac{1}{n}\int_0^x \frac{(x-u)^{n-2}}{(n-2)!} \sum_{j=1}^n f_j^r(u)\, du\right\}^{\frac{n}{r}}.$$

Now by the arithmetic-geometric mean inequality,

$$\frac{1}{n}\sum_{j=1}^n f_j^r(u) \geqq \left(\prod_{j=1}^n f_j(u)\right)^{\frac{r}{n}},$$

Hence, using [2, Theorem 184] with $p(u)=(x-u)^{n-2}/(n-2)!$,

$$\left\{\int_0^x p(u)\frac{1}{n}\sum_{j=1}^{n} f_j^r(u)\,du\right\}\geqslant\left\{\int_0^x p(u)\left[\left\{\prod_{j=1}^{n} f_j(u)\right\}^{\frac{1}{n}}\right]^r du\right\}^{\frac{n}{r}}$$

$$\geqslant\left(\int_0^x p(u)\,du\right)^{\frac{n}{r}}\exp\left\{\frac{n\int_0^x p(u)\frac{1}{n}\log\left\{\prod_{j=1}^{n} f_j(u)\right\}du}{\int_0^x p(u)\,du}\right\}.$$

Finally we note that $\int_0^x p(u)\,du=x^{n-1}/(n-1)!=I_n(x)$, so that on combining the last inequality with (10) we obtain

$$f_1*f_2*\cdots*f_n(x)\geqslant I_n(x)\exp\left\{\frac{\int_0^x\frac{(x-u)^{n-2}}{(n-2)!}\log\left\{\prod_{j=1}^{n} f_j(u)\right\}du}{x^{n-1}/(n-1)!}\right\},$$

which reduces to (8), and completes the proof of the theorem.

The fact that the inequality (3) is weaker than (1) for the class of symmetric, nonnegative kernels on $[0, a]\times[0, a]=A_2$ follows from [2, Theorem 184] which gives

$$\int_0^a\int_0^a K\,dx\,dy\geqslant a^2\exp\left\{\int_0^a\int_0^a \log K\,dx\,dy/a^2\right\}.$$

We conclude this paper by noting that, for this class of kernels, (1) is actually valid for all $n\geqslant 1$. More generally, if v is any nonnegative function (for simplicity, we assume v and K are continuous), then

$$\text{(11)}\qquad \left(\int_0^a v^2(x)\,dx\right)^{n-1}\int_0^a\int_0^a v(x)\,v(y)\,K_n(x, y)\,dx\,dy$$
$$\geqslant\left(\int_0^a\int_0^a v(x)\,v(y)\,K(x, y)\,dx\,dy\right)^n,$$

for all $n\geqslant 1$; setting $v(x)=1$, (1) is obtained. The inequality (11) follows at once from the matrix inequality $(v_T v)^{n-1}(v_T A^n v)\geqslant(v_T A v)^n$, or

$$\text{(12)}\qquad \left(\sum_{i=1}^{k} v_i^2\right)^{n-1}\sum_{i=1}^{k}\sum_{j=1}^{k} v_i\,a_{ij}^{(n)}v_j\geqslant\left(\sum_{i=1}^{k}\sum_{j=1}^{k} v_i\,a_{ij}\,v_j\right)^n$$

proved by MULHOLLAND and SMITH [4], by replacing the integrals in (11) by approximating RIEMANN sums of the form appearing in (12), then taking limits as $k\to\infty$. In (12), the matrix $A=(a_{ij})$ is a $k\times k$ symmetric matrix with all $a_{ij}\geqslant 0$, and $v_T=(v_1, \ldots, v_k)$ has all $v_i\geqslant 0$. In the paper [1] the authors mentioned the inequality (12) in a note (added in proof), but apparently did not notice that (12) implied (11), and hence also (1).

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