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253. A COMPARISON OF TWO TRIGONOMETRIC INEQUALITIES*

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In a triangle T, a, b, c are the sides opposite the vertices A, B, C and α, β, γ are the angles at A, B, C. The altitudes issuing from A, B, C are denoted by h_a, h_b, h_c, the inradius by r, the circumradius by R, and the semiperimeter by s. H denotes the orthocenter and I the incenter of T.

1. From the well-known relation

$$IH^2 = 2r^2 - 4R^2 \cos\alpha\cos\beta\cos\gamma = 2r^2 - 4R^2\left[\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma - 2}{2}\right] \geqslant 0$$

we readily obtain

$$(1) \qquad 2 + \frac{r^2}{R^2} \geqslant \sin^2\alpha + \sin^2\beta + \sin^2\gamma,$$

thus verifying the inequality

$$\cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \equiv 2 + \frac{r}{R} \geqslant 2 + \frac{r^2}{R^2} \geqslant \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$$

2. In Problem E 1272 (Amer. Math. Monthly, **64** (1957), 432; **65** (1958), 123; **67** (1960), 693), it was shown that

$$\cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \geqslant \left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)^2.$$

3. The purpose of this note is to show that the latter inequality is stronger than the former, namely that

$$\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)^2 \geqslant \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$$

Proof. From the well-known identity

$$ab + bc + ca = r^2 + s^2 + 4Rr = r^2 + \left(\frac{a^2 + b^2 + c^2}{4} + \frac{ab + bc + ca}{2}\right) + 4Rr,$$

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we obtain

$$ab+bc+ca=2r^2+\frac{a^2+b^2+c^2}{2}+8Rr.$$

Then

$$h_a+h_b+h_c=\frac{ab+bc+ca}{2R}=\frac{r^2}{R}+R(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)+4r.$$

But

$$AI+BI+CI+3r \geqslant h_a+h_b+h_c.$$

Hence

$$AI+BI+CI \geqslant \frac{r^2}{R}+R(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)+r,$$

or

$$2\left(\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$\geqslant \frac{r^2}{2R^2}+\frac{\sin^2\alpha+\sin^2\beta+\sin^2\gamma}{2}+\frac{r}{2R}.$$

From the inequality (1) above, we obtain

$$2\left(\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$\geqslant \sin^2\alpha+\sin^2\beta+\sin^2\gamma-\left(1-\frac{r}{2R}\right)$$

and, since $\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}=1-\frac{r}{2R}$, we have

$$\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)^2 \geqslant \sin^2\alpha+\sin^2\beta+\sin^2\gamma.$$