PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 247 — № 273 (1969)

251. SOME INEQUALITIES FOR TRIANGLE*

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In this paper we will prove some inequallties related to the elements of triangle. Some of them are generalisations of already known results and some are new.

We use the following notations:

a, b, c for the sides of the triangle:

F the area of the triangle;

R the radius of the circumscribed circle of the triangle;

r the radius of the inscribed circle of the triangle;

r_a, r_b, r_c the radii of the exscribed circles corresponding to the sides a, b, c;

w_a, w_b, w_c bisectors of the angles of the triangle corresponding to the sides a, b, c;

h_a, h_b, h_c the altitudes of the triangle corresponding to the sides a, b, c.

1. J. ANDERSSON (see [1]) has proved the inequalty

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\leqslant\frac{1}{2}\,\frac{abc}{r}, \tag{1.1}$$

with equality if and only if the triangle is equilateral.

Inequality (1.1) is equivalent to the following

$$a^3(s-a)+b^3(s-b)+c^3(s-c)\leqslant abcs, \tag{1.2}$$

where $2s=a+b+c$ and with the equality if and only if the triangle is equilateral.

We first, prove, the following

Theorem 1.1. *If λ is real, then*

$$a^\lambda(s-a)+b^\lambda(s-b)+c^\lambda(s-c)\leqslant\frac{1}{2}\,abc\,(a^{\lambda-2}+b^{\lambda-2}+c^{\lambda-2}), \tag{1.3}$$

with equality holding only for the equilateral triangle.

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* Presented January 5, 1969 by D. S. Mitrinović and R. R. Janić.

Proof. Since

$$a^{\lambda}(s-a)+b^{\lambda}(s-b)+c^{\lambda}(s-c)$$
$$=\frac{1}{2}\{-a^{\lambda+1}-b^{\lambda+1}-c^{\lambda+1}+a^{\lambda}(b+c)+b^{\lambda}(c+a)+c^{\lambda}(a+b)\},$$

and using SCHUR's inequality (see, for example [2])

$$a^{\lambda-1}(a-b)(a-c)+b^{\lambda-1}(b-c)(b-a)+c^{\lambda-1}(c-a)(c-b)$$
$$=a^{\lambda+1}+b^{\lambda+1}+c^{\lambda+1}-a^{\lambda}(b+c)-b^{\lambda}(c+a)-c^{\lambda}(a+b)+abc(a^{\lambda-2}+b^{\lambda-2}+c^{\lambda-2})\geqslant 0,$$

it immediately follows that (1.3) holds.

Since SCHUR's inequality is reduced to equality if and only if $a=b=c$, (1.3) is reduced to equality only for equilateral triangles.

This concludes the proof.

Note that inequality (1.3) is more general than (1.2) or what is the same (1.1), and is reduced to the latter for $\lambda=3$.

Now we prove two other inequalities for the above listed elements of triangle.

Theorem 1.2. *If λ is real, then*

$$a^{\lambda}w_a+b^{\lambda}w_b+c^{\lambda}w_c\leqslant\sqrt{\frac{3}{2}abcs\,(a^{2(\lambda-2)}+b^{2(\lambda-2)}+c^{2(\lambda-2)})} \tag{1.4}$$

with equality if and only if the triangle is equilateral.

Proof. Since (see, for example [3])

$$w_a\leqslant\sqrt{s(s-a)},\quad w_b\leqslant\sqrt{s(s-b)},\quad w_c\leqslant\sqrt{s(s-c)}, \tag{1.5}$$

we obtain

$$a^{\lambda}w_a+b^{\lambda}w_b+c^{\lambda}w_c\leqslant a^{\lambda}\sqrt{s(s-a)}+b^{\lambda}\sqrt{s(s-b)}+c^{\lambda}\sqrt{s(s-c)} \tag{1.6}$$
$$\leqslant\sqrt{3s\,(a^{2\lambda}(s-a)+b^{2\lambda}(s-b)+c^{2\lambda}(s-c))}.$$

Taking

$$x=a^{\lambda}\sqrt{s(s-a)},\quad y=b^{\lambda}\sqrt{s(s-b)},\quad z=c^{\lambda}\sqrt{s(s-c)},$$

and applying the inequality

$$x+y+z\leqslant(3(x^2+y^2+z^2))^{\frac{1}{2}}\qquad(x,y,z>0),$$

we have by Theorem 1.1,

$$a^{2\lambda}(s-a)+b^{2\lambda}(s-b)+c^{2\lambda}(s-c)\leqslant\frac{1}{2}abc\,(a^{2(\lambda-1)}+b^{2(\lambda-1)}+c^{2(\lambda-1)}), \tag{1.7}$$

and (1.6) is reduced to (1.4).

Since all of the inequalities (1.5), (1.6), (1.7) become equalities if and only if $a=b=c$ it follows that (1.4) is also reduced to an equality if and only if $a=b=c$.

Theorem 1.3. *If λ is real, than*

$$a^{\lambda}w_a^2+b^{\lambda}w_b^2+c^{\lambda}w_c^2\leqslant\frac{1}{2}abcs\,(a^{\lambda-2}+b^{\lambda-2}+c^{\lambda-2}), \tag{1.8}$$

with equality if and only if the triangle is equilateral.

Proof. Using inequality (1.5) and Theorem 1.1, we get

$$a^\lambda w_a^2 + b^\lambda w_b^2 + c^\lambda w_c^2 \leqslant s\,(a^\lambda (s-a) + b^\lambda (s-b) + c^\lambda (s-c))$$
$$\leqslant \frac{1}{2} abcs\,(a^{\lambda-2} + b^{\lambda-2} + c^{\lambda-2}).$$

Since (1.5) and (1.4) become equalities if and only if $a=b=c$, (1.8) is equality if and only if the triangle is equilateral, what was to be proved.

Theorem 1.4. *For each triangle, the inequality*

$$(1.9)\qquad 12F \leqslant a\,(h_b+h_c) + b\,(h_c+h_a) + c\,(h_a+h_b) < \frac{5}{2}(a^2+b^2+c^2) + \frac{2F^2}{Rr}$$

holds, with equality holding only for the equilateral triangle.

Proof. Since $h_b+h_c \geqslant 2\sqrt{h_b h_c}$, $h_c+h_a \geqslant 2\sqrt{h_c h_a}$, $h_a+h_b \geqslant 2\sqrt{h_a h_b}$, we get

$$a\,(h_b+h_c) + b\,(h_c+h_a) + c\,(h_a+h_b) \geqslant 2\,(a\sqrt{h_b h_c} + b\sqrt{h_c h_a} + c\sqrt{h_a h_b})$$
$$\geqslant 6\sqrt[3]{abch_a h_b h_c}$$
$$= 12F,$$

with equality if and only if $h_a=h_b=h_c$, that is for equilateral triangles.

This proves the first part of (1.9).

Using the relations between arithmetic and geometric means, we obtain

$$a\,(h_b+h_c) \leqslant \left(\frac{a+h_b+h_c}{2}\right)^2,\quad b\,(h_c+h_a) \leqslant \left(\frac{b+h_c+h_a}{2}\right)^2,\quad c\,(h_a+h_b) \leqslant \left(\frac{c+h_a+h_b}{2}\right)^2.$$

At most one of these inequalities may be reduced to equality for otherwise there would exist a triangle with two obtuse angles.

Hence

$$a\,(h_b+h_c) + b\,(h_c+h_a) + c\,(h_a+h_b)$$
$$< \frac{1}{4}\{a^2+b^2+c^2+2\,(h_a^2+h_b^2+h_c^2)+2\,(h_b h_c+h_c h_a+h_a h_b)$$
$$+2\,(a\,(h_b+h_c)+b\,(h_c+h_a)+c\,(h_a+h_b))\},$$

i.e.

$$a\,(h_b+h_c) + b\,(h_c+h_a) + c\,(h_a+h_b)$$
$$< \frac{1}{2}\{a^2+b^2+c^2+2\,(h_a^2+h_b^2+h_c^2)+2\,(h_b h_c+h_c h_a+h_a h_b)\}.$$

Since (see [3])

$$h_a^2+h_b^2+h_c^2 \leqslant \frac{3}{4}(a^2+b^2+c^2) \quad\text{and}\quad h_b h_c+h_c h_a+h_a h_b = \frac{2F^2}{Rr},$$

we finally obtain

$$a\,(h_b+h_c) + b\,(h_c+h_a) + c\,(h_a+h_b) < \frac{5}{2}(a^2+b^2+c^2) + \frac{2F^2}{Rr},$$

what proves the last statement.

2. Let x, y, z be nonnegative real numbers. The mean of the order k is definied by

$$M_k(x, y, z) = \left(\frac{x^k + y^k + z^k}{3}\right)^{\frac{1}{k}} \qquad (k \neq 0 \text{ and } |k| < +\infty),$$

$$= \sqrt[3]{xyz} \qquad (k = 0).$$

We prove the following theorems:

Theorem 2.1. *For elements of a triangle inequalities*

(2.1) $$M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) \geqslant \sqrt[3]{\frac{4r^2}{27R^2}} \qquad (k > 0),$$

(2.2) $$M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + b_b}, \frac{h_c - r}{h_c + r_c}\right) \leqslant \sqrt[3]{\frac{2r}{27R}} \qquad (k < 0),$$

(2.3) $$\sqrt[3]{\frac{4r^2}{27R^2}} \leqslant M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) \leqslant \sqrt[3]{\frac{2r}{27R}} \qquad (k = 0),$$

hold.

Equalities hold if and only if the triangle is equilateral.

Proof. By

$$2F = ah_a = 2sr, \quad 2F = ah_a = 2(s - a)r_a \qquad (2s = a + b + c)$$

we have

(2.4) $$a(h_a - r) = (b + c)r, \quad a(h_a + r_a) = (b + c)r_a.$$

By (2.4) we obtain

(2.5) $$\frac{h_a - r}{h_a + r_a} = \frac{r}{r_a}.$$

Similarly we obtain equalities

(2.6) $$\frac{h_b - r}{h_b + r_b} = \frac{r}{r_b}, \quad \frac{h_c - r}{h_c + r_c} = \frac{r}{r_c}.$$

From (2.5) and (2.6) we get

(2.7) $$M_0\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) = \frac{r}{\sqrt[3]{r_a r_b r_c}}.$$

By (see [4])

(2.8) $$3\sqrt[3]{2^{-1}}\sqrt[3]{r^2}\sqrt[3]{R} \leqslant \sqrt[3]{r_a r_b r_c} \leqslant 3\sqrt[3]{2^{-2}}\sqrt[3]{r}\sqrt[3]{R^2}$$

we obtain inequality (2.3). Since in (2.8) equality holds if and only if the triangle is equilateral, then in (2.3) equality also holds if and only if the triangle is equilateral.

Inequalities (2.1) and (2.2) are obtained by (2.3) and by the fact that M_k is a monotonic function for k. Since equality $M_0(x, y, z) = M_k(x, y, z)$ $(k \neq 0)$ holds if and only if $x = y = z$, in (2.1) and (2.2) equality holds if and only if the triangle is equilateral.

Remark. Inequalities (2.1), (2.2), (2.3) hold also if in fractions, denominators permute in an arbitrary manner.

The proof is completely identical to the proof of Theorem 2.1.

Theorem 2.2. *If* $a \neq b \neq c \neq a$, *inequalities*

(2.9) $$M_k\left(\frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c}\right) < 1 \qquad (k < 0),$$

(2.10) $$M_k\left(\frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c}\right) > 1 \qquad (k > 0)$$

hold.

For $k = 0$ *equality*

(2.11) $$M_k\left(\frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c}\right) = 1$$

holds.

Proof. By equalities

$$2F = ah_a = (c + a - b)\, r_b \text{ and } 2F = ah_a = (a + b - c)\, r_c$$

we obtain

(2.12) $$a(h_a - r_b) = (c - b)\, r_b \text{ and } a(r_c - h_a) = (c - b)\, r_c.$$

Since, by hypothesis, $a \neq b \neq c \neq a$ and $r_c \neq h_a$, by (2.12) we have

(2.13) $$\frac{h_a - r_b}{r_c - h_a} = \frac{r_b}{r_c}.$$

Similarly

(2.14) $$\frac{h_b - r_c}{r_a - h_b} = \frac{r_c}{r_a} \text{ and } \frac{h_c - r_a}{r_b - h_c} = \frac{r_a}{r_b}.$$

By (2.13) and (2.14) we obtain directly (2.11).

Since M_k is a monotonic increasing function for k, inequalites (2.9) and (2.10) follow from (2.11).

This paper in manuscript form was read by P. M. Vasić and was returned to me with helpful suggestions for which I thank him.

REFERENCES

[1] J. Andersson, *Problem* 966, Amer. Math. Monthly **59** (1952), 41 and **59** (1952), 466.

[2] D. S. Mitrinović, *Nejednakosti*, Beograd 1954.

[3] D. S. Mitrinović, P. M. Vasić, R. Ž. Đorđević, R. R. Janić, *Geometrijske nejednakosti*, Matematička biblioteka, sv. 31, Beograd 1966.

[4] A. Makowski, *Inequalities for radii of inscribed, circumscribed and escribed circles*, Nieuw Arch. Wisk. (3) **12** (1964), 5—7.