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249. AN INEQUALITY DUE TO HENRICI*

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1. The main purpose of this note is to show that an inequality due to P. HENRICI, [7, (5)] and several generalisations due to MITRINOVIĆ and VASIĆ, [7], are consequences of a general inequality obtained in [4]. First we give a more general form of the main result in [4]; the generalisation consists of allowing different weights in the means involved.

In what follows $b=\{b_1, b_2, \dots\}$, $p=\{p_1, p_2, \dots\}$, and $q=\{q_1, q_2, \dots\}$ will be sequences of positive numbers; write $Q_n=\sum_{k=1}^{n} q_k$. Functions will always mean real valued functions of a real variable. If f is a function $a=\{a_1, a_2, \dots\}$ any sequence of real numbers then $f(a)$ will denote the sequence $\{f(a_1), f(a_2), \dots\}$. The following notations are standard;

$$M_n^{[r]}(a; q)=\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k^{\,r}\right)^{1/r}, \quad r\neq 0, \quad |r|<\infty,$$

$$A_n(a; q)=M_n^{[1]}(a; q),$$

$$G_n(a; q)=\left(\prod_{k=1}^{n} a_k^{\,q_k}\right)^{1/Q_n}=M_n^{[0]}(a; q).$$

If possible a is allowed to take negative or zero values; this is essential as we wish to apply our main result to sequences of the form $f(a)$ which may not be positive even if a is.

2. Theorem 1. *Let F, G be two functions such (a) F is strictly monotonic, (b) G is concave, (c) $G\circ F$ is convex, then*

$$(1) \qquad Q_n\left\{G(A_n(a; q))-G\circ F\left(\frac{q_{n+1}Q_n}{p_{n+1}Q_n}A_n(F^{-1}(a); p)\right)\right\}$$

$$\leqslant Q_{n+1}\left\{G(A_{n+1}(a; q)-G\circ F\left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(F^{-1}(a); p)\right)\right\}.$$

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Further if G is strictly concave and $G\circ F$ strictly convex then inequality (1) *is strict unless*

(i) $$G_{n+1}=A_n(a;\,q),$$

(ii) $$G_{n+1}=F\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(F^{-1}(a);\,p)\right).$$

If G is a linear and $G\circ F$ is strictly convex then inequality (1) *is strict unless (ii) holds; whereas if G is strictly concave and $G\circ F$ is linear, inequality* (1) *is strict unless (i) holds. If convex and concave are interchanged the inequality* (1) *is reversed.*

Proof. Since the proof is essentially the same as the proof of Theorem 2 in [**4**] only a few details will be given.

If $B_n=\dfrac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(F^{-1}(a);\,p)$ and $B_{n+1}=\dfrac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(F^{-1}(a);\,p)$ then it is easily seen that

(2) $$Q_n(B_n-F^{-1}(a_{n+1}))=Q_{n+1}(B_{n+1}-F^{-1}(a_{n+1})).$$

Further the difference between the righthand side and the lefthand side of (1) is equal to

(3) $$Q_{n+1}G(A_{n+1}(a;\,q))-Q_nG(A_n(a:\,q))-Q_{n+1}G\cdot F(B_{n+1})+Q_nG\cdot F(B_n).$$

Since (2), and (3) are identical to the first two lines of the proof of Theorem 2, [**4**], the rest follows as in that paper.

Remark. By saying a function is linear we mean that it is both convex and concave.

Corollary 2. *If F is strictly monotonic and convex then*

(4) $$Q_n\left\{A_n(F(a);\,q)-F\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(a;\,p)\right)\right\}\le Q_{n+1}\left\{A_{n+1}(F(a);\,q)-F\left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(a;\,p)\right)\right\},$$

and if F is strictly convex then inequality (4) *is strict unless* $a_{n+1}=\dfrac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(a;\,p)$.

If convex is replaced by concave inequality (4) *is reversed.*

In particular we have

(5) (a) $$Q_n\{A_n(b;\,q)-G_n(b;\,p)^{q_{n+1}P_n/p_{n+1}Q_n}\}\le Q_{n+1}\{A_{n+1}(b;\,q)-G_{n+1}(b;\,p)^{q_{n+1}P_{n+1}/p_{n+1}Q_{n+1}}\},$$

with equality only when $b_{n+1}^{p_{n+1}/q_{n+1}}=(G_n(b;\,p))^{P_n/Q_n}$;

(6) (b) $$\left(\frac{A_n(b;\,p)\,P_n}{G_n(b;\,q)\,Q_n}\right)^{Q_n}\le\left(\frac{q_{n+1}}{p_{n+1}}\right)^{q_{n+1}}\left(\frac{A_{n+1}(b;\,p)\,P_{n+1}}{G_{n+1}(b;\,q)\,Q_{n+1}}\right)^{Q_{n+1}},$$

with equality only when $b_{n+1}=\dfrac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(b;\,p)$,

(c) *if* $\frac{r}{s} \geqslant 1$ *or* $\frac{r}{s} \leqslant 0$ *then*

$$(7)\quad Q_n\left\{(M_n^{[r]}(b;q))^r - \left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}\right)^{r/s}(M_n^{[s]}(b;p))^r\right\}$$
$$\leqslant Q_{n+1}\left\{M_{n+1}^{[r]}(b;q) - \left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}\right)^{r/s}(M_{n+1}^{[s]}(b;p))^r\right\},$$

with equality only when $r=s$, $r=0$ *or neither of these hold and*

$$b_{n+1} = \frac{q_{n+1}P_n}{p_{n+1}Q_n} M_n^{[s]}(b;p).$$

Proof. If Theorem 1 is applied with $G=1$ and the sequence $F^{-1}(a)$, inequality (1) reduces to inequality (5) and the cases of equality are those of the second possibility in Theorem 1.

(a) follows by taking $F(x)=e^x$ and using the sequence $a=\log b$.

For (b) take $F(x)=\log x$ and $a=b$.

Finally if $F(x)=x^{r/s}$, $a=b^s$ then we get (c).

Remarks. Inequality (4) generalises an inequality in **[4]**, and so is a generalisation of results in **[8]**. Inequalities (5) and (6) are just (2.1) of **[1]** and (1.2) of **[1]**, or (2.2) of **[3]**, respectively, where different proofs are given; inequality (6) is due to Mitrinović and Vasić, **[5]**. Finally (c) is the case $\lambda=1$ of (3.1) in **[2]**, which is due to Mitrinović and Vasić, **[6]**.

Corollary 3. *If G is strictly monotonic and concave then*

$$(8)\quad Q_n\left\{G(A_n(a;q)) - \frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(G(a);p)\right\}$$
$$\leqslant Q_{n+1}\left\{G(A_{n+1}(a;q)) - \frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(G(a);p)\right\};$$

and if G is strictly concave then inequality (8) *is strict unless* $G_{n+1}=A_n(a;q)$. *If concave is replaced by convex inequality* (8) *is reversed.*

In particular we have

$$(9)\quad (a)\quad \frac{(A_n(b;q))^{Q_n/q_{n+1}}}{(G_n(b;p))^{P_n/p_{n+1}}} \leqslant \frac{(A_{n+1}(b;q))^{Q_{n+1}/q_{n+1}}}{(G_{n+1}(b;p))^{P_{n+1}/p_{n+1}}},$$

with equality only when $a_{n+1}=A_n(a;q)$;

$$(10)\quad (b)\quad Q_n\left\{\frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(b;p) - G_n(b;q)\right\}$$
$$\leqslant Q_{n+1}\left\{\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(b;p) - G_{n+1}(b;q)\right\},$$

with equality only when $b_{n+1}=G_n(b;q)$;

(c) *if* $r/s \leq 1$, *then*

$$(11)\qquad Q_n\left\{(M_n^{[s]}(b;q))^s - \frac{q_{n+1}P_n}{p_{n+1}Q_n}(M_n^{[r]}(b;p))^s\right\} \leq Q_{n+1}\left\{(M_{n+1}^{[s]}(b;q))^s - \frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}(M_{n+1}^{[r]}(b;p))^s\right\},$$

with equality only if $r=s$ *or* $r\neq s$ *and* $b_{n+1}=M_n^{[r]}(b;p)$.

Proof. Take, in Theorem 1, $G\circ F=1$, then inequality (1) reduces to inequality (8) and the cases of equality are those of the third possibility in Theorem 1.

(a) then follows by taking $G(x)=\log x$, and $a=b$.

For (b) take $G(x)=e^x$ and $a=\log b$, and for (c) let $a=b^s$, $G(x)=x^{r/s}$.

Remarks. Again, inequality (8) generalises one in [4], and so also one in [8]. Inequality (9) is the case $\lambda=0$ of Theorem 5 in [1]. (There are two misprints in this theorem; firstly we should have $\lambda\geq 0$ and secondly the case of equality should read $A_{n-1}(a;q)=a_n+\lambda$.) Inequality (10) is the case $\lambda=1$ of Theorem 4 of [1] and is due to MITRINOVIĆ and VASIĆ, [6]. (There is a misprint in Theorem 4 of [1]; the case of equality should read $\lambda G_{n-1}(a;p)=a_n$.) Finally inequality (11) is the case $\lambda=1$ of Theorem 4 of [2] and is due to MITRINOVIĆ and VASIĆ, [6].

The inequalities above derive from (1) by only allowing one non-trivial function to occur; there are many generalisations if we allow two functions in (1). Thus if $G(x)=x^\mu$, $G\circ F(x)=x^\lambda$, $0\leq\mu\leq 1$, $\lambda\geq 1$ or $\lambda<1$ and if we put $a=b^\nu$, $\nu\neq 0$ we get from (1) that

$$(12)\qquad Q_n\left\{(M_n^{[\nu]}(b;q))^{\mu\nu} - \left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}\right)^\lambda\left(M_n^{\left[\frac{\mu\nu}{\lambda}\right]}(b;p)\right)^{\mu\nu}\right\} \leq Q_{n+1}\left\{(M_{n+1}^{[\nu]}(b;q))^{\mu\nu} - \left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}\right)^\lambda\left(M_{n+1}^{\left[\frac{\mu\nu}{\lambda}\right]}(b;p)\right)^{\mu\nu}\right\},$$

with equality occcurring as follows; (α) if $\mu\neq 0$, 1, $\lambda\neq 1$ then inequality (12) is strict unless

$$\text{(i)}\quad b_{n+1}=M_n^{[\nu]}(b;q) \qquad\text{and}\qquad \text{(ii)}\quad b_{n+1}=\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}\right)^{\frac{\lambda}{\mu\nu}} M_n^{\left[\frac{\mu\nu}{\nu}\right]}(b;p);$$

(β) if $\mu=0$ or 1, $\lambda\neq 1$ then inequality (12) is strict unless (ii) holds; (γ) if $\mu\neq 0$, 1, $\lambda=1$ then inequality (12) is strict unless (i) holds.

If $\lambda=1$ this inequality (12) reduces to (11) and is typical of many possible generalisations of the inequalities (5)—(7), (9)—(11) that can be derived from Theorem 1.

3. Corollary 4. *If* $b_{n+1}\geq 1$ *and* $G_n(b;p)\geq 1$ *then*

$$(13)\qquad \left\{\sum_{k=1}^{n}\frac{q_k}{1+b_k} - \frac{Q_n}{1+(G_n(b;p))^{\frac{q_{n+1}P_n}{p_{n+1}Q_n}}}\right\} \leq \left\{\sum_{k=1}^{n+1}\frac{q_k}{1+b_k} - \frac{Q_n}{1+(G_{n+1}(b;p))^{\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}}}\right\},$$

with equality iff $b_{n+1}=(G_n(b;p))^{\frac{q_{n+1}P_n}{p_{n+1}Q_n}}$. *In particular if* $b_k\geqslant 1$, $1\leqslant k\leqslant n$,

$$\sum_{k=1}^{n}\frac{q_k}{1+b_k}\geqslant\frac{Q_n}{1+(G_n(b;p))^{\frac{q_nP_n}{p_nQ_n}}}+\frac{q_1(b_1^{\frac{p_1q_n}{p_nq_1}}-b_1)}{(1+b_1)(1+b_1^{\frac{p_1q_n}{p_nq_1}})}. \tag{14}$$

Proof. Apply Theorem 1 to the sequence $a=f(b)$, with $G=1$, $F=f\circ g$ where $f(x)=1/(1+x)$ and $g(x)=e^x$. Then if $x\geqslant 0$, F is strictly convex and Theorem 1 can be applied provided the function $G\circ F$ is only used in this range in the proof of Theorem 1 (i.e. if B_n, B_{n+1} and a_{n+1} are non-negative).

Remarks. Inequality (13) is a generalisation of Proposition 2 of [7] and inequality (14) is a generalisation of HENRICI's inequality, [7, Proposition 3].

Corollary 5. *If* $r, s\neq 0$, $s/r\geqslant 1$, $b_{n+1}^s\geqslant\frac{s-r}{s+r}$, $\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}\right)^{s/r}(M_n^{[r]}(b;p))^s\geqslant\frac{s-r}{s+r}$, *then*

$$\left\{\sum_{k=1}^{n}\frac{q_k}{1+b_k^s}-\frac{Q_n}{1+\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}\right)^{s/r}(M_n^{[r]}(b;p))^s}\right\} \tag{15}$$
$$\leqslant\left\{\sum_{k=1}^{n+1}\frac{q_k}{1+b_k^s}-\frac{Q_{n+1}}{1+\left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}\right)^{s/r}(M_{n+1}^{[r]}(b;p))^s}\right\},$$

with equality iff $b_{n+1}=\frac{q_{n+1}P_n}{p_{n+1}Q_n}M_n^{[r]}(b;p)$. *In particular if* $b_k^s\geqslant\frac{s-r}{s+r}$, $1\leqslant k\leqslant n$,

$$\sum_{k=1}^{n}\frac{q_k}{1+b_k^s}\geqslant\frac{Q_n}{1+\left(\frac{q_nP_n}{p_nQ_n}\right)^{s/r}(M_n^{[r]}(b;p))^s}+\frac{q_1}{1+b_1^s}-\frac{q_1}{1+\left(\frac{q_np_1}{p_nq_1}\right)^{s/r}b_1^s}. \tag{16}$$

Proof. The same as Corollary 4 but with $f(x)=\frac{1}{1+x^s}$, $g(x)=x^{\frac{1}{r}}$.

In particular if $p_k=q_k=1$, $1\leqslant k\leqslant n$, (16) reduces to

$$\sum_{k=1}^{n}\frac{1}{1+b_k^s}\geqslant\frac{n}{1+(M_n^{[r]}(b))^s}, \tag{17}$$

an inequality similar to HENRICI's inequality.

A simple direct proof of HENRICI's inequality and of (17) can be given as follows.

Theorem 6. *If* $f(x)=1/(1+\alpha(x))$ *is convex for* x *in the set* E, $\alpha>0$, *then*

$$\sum_{k=1}^{n}\frac{q_k}{1+\alpha(a_k)}-\frac{Q_n}{1+\alpha\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(a;p)\right)} \tag{18}$$
$$\leqslant\sum_{k=1}^{n+1}\frac{q_k}{1+\alpha(a_k)}-\frac{Q_{n+1}}{1+\alpha\left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(a;p)\right)}$$

provided a_{n+1}, B_n *and* B_{n+1} *are in the set* E. (*Here* B_n, B_{n+1} *are as in Theorem* 1 *with* $F=1$.)

Proof. Simple calculations show that (18) is equivalent to

$$\frac{q_{n+1}}{Q_{n+1}}\,\frac{1}{1+\alpha(a_{n+1})}+\frac{Q_n}{Q_{n+1}}\,\frac{1}{1+\alpha(B_n)}\geqslant\frac{1}{1+\alpha(B_{n+1})}. \tag{19}$$

But (19) is immediate from the convexity of f, provided a_{n+1}, B_n, B_{n+1} are in E.

Particular choices of α will give inequalities (13) and (15); and of course inequality (18) can be deduced from Theorem 1.

4. Theorem 1 can be generalised further as follows; if λ is any real number then

$$Q_n\left\{G(A_n(a;\,q))-G\circ F\left(\frac{q_{n+1}P_n}{p_{n+1}Q_n}A_n(F^{-1}(a);\,p)+\lambda\frac{Q_{n+1}}{Q_n}\right)\right\} \tag{20}$$
$$\leqslant Q_{n+1}\left\{G(A_{n+1}(a;\,q)-G\circ F\left(\frac{q_{n+1}P_{n+1}}{p_{n+1}Q_{n+1}}A_{n+1}(F^{-1}(a);\,p)+\lambda\right)\right\}.$$

This in turn leads to generalisations of all the other inequalities above; in particular we can get Theorems 4 and 5 of **[1]**, of which inequalities (8) and (9) are particular cases.

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