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235. ON THE DIOPHANTINE EQUATION

$$x^3+y^3-z^3=px+py-qz^*$$

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1. It is known that the DIOPHANTINE equation

$$x^3-x+y^3-y=z^3-z, \tag{1}$$

which expresses the condition that one tetrahedral number be the sum of two tetrahedral numbers has infinitely many solutions in integers and indeed infinitely many in which x, y, z are natural numbers. (See **[1]**, **[2]**).

So far as I am aware no parametric solutions have been published so that the polynomial solutions, incomplete though they be, given in Theorem 1 may have some interest. These parametric solutions lead also to parametric solutions of such DIOPHANTINE equations as

$$x^3+y^3=z^3-z, \quad x^3+y^3=z^3+z, \quad x^3-x+y^3-y=z^3 \tag{2}$$

which are particular cases of the equation

$$x^3+y^3-z^3=p(x+y)-qz. \tag{3}$$

For certain pairs of values of the integers p, q it is possible to deduce parametric solutions by means of those for $p=q=1$. On the other hand I have not been able to find for the equation

$$x^3-x+y^3+y=z^3 \tag{4}$$

any non-trivial sulution apart from $(10, 9, 12)$, $(-10, -9, -12)$. (Equation (4) has obviously three sets of trivial solutions corresponding to: (*i*) $x=z$, (*ii*) $y=z$, (*iii*) $x=-y$ respectively.)

2. Theorem 1. *Let h be any integer, r any non-negative integer. Define θ by the equations*

$$\cosh\theta=\frac{1}{2}(h-1)(3h+3)^{\frac{1}{2}}, \quad \sinh\theta=\frac{1}{2}\{4h^3-(h+1)^3\}^{\frac{1}{2}} \tag{5}$$

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* Presented May 6. 1968 by D. S. Mitrinović.

(*so that* $\cosh^2\theta - \sinh^2\theta = 1$). *Define* M, N *by the equaticns*

$$(6) \qquad N = (h-1)\frac{\cosh(2r+1)\theta}{\cosh\theta}, \quad M = \frac{\sinh(2r+1)\theta}{\sinh\theta};$$

define x, y, z *by the relations*

$$(7) \qquad x = \frac{1}{2}(h+1)M + \frac{1}{2}N, \quad y = \frac{1}{2}(h+1)M - \frac{1}{2}N, \quad z = hM.$$

Then x, y, z *are rational integers such that*

$$x^3 - x + y^3 - y = z^3 - z.$$

Proof. Since $4\cosh^2\theta$ is a raticnal integer and since $\cosh(2r+1)\theta/\cosh\theta$ is an integral polynomial in $4\cosh^2\theta$ it follows that N is a rational integer. Since $4\sinh^2\theta$ is a raticnal integer and since $\sinh(2r+1)\theta/\sinh\theta$ is an integral polynomial in $4\sinh^2\theta$, it follows that M is a rational integer.

Now $(h+1)M$ and N are both odd or both even: hence x, y given by (7) are rational integers. That z is integral is plain.

Next, the definitions of M, N show that

$$(3h+3)^{\frac{1}{2}}N \pm \{4h^3-(h+1)^3\}^{\frac{1}{2}}M = 2\{\cosh(2r+1)\theta \pm \sinh(2r+1)\theta\}$$

so that the integers N, M satisfy the equation

$$(8) \qquad (3h+3)N^2 - \{4h^3-(h+1)^3\}M^2 = 4.$$

But now the identity

$$(9) \qquad 4[x^3+y^3-z^3-x-y+z] = M[(3h+3)N^2-\{4h^3-(h+1)^3\}M^2-4]$$

shows that the integers x, y, z as defined in (7) do in fact satisfy the DIOPHANTINE equation $x^3+y^3-z^3 = x+y-z$. Theorem 1 is proved.

3. The special case $r=0$ gives $x=h$, $y=1$, $z=h$; a „trivail" solution but one which nevertheless gives rise (for $|h|\geqslant 2$) to the parametric solution described. (This trivial solution was used in [2] to obtain equation (8) with infinitely many integral solutions M, N (in which $(h+1)M, N$ have the same parity) and hence (by (9)) infinitely many non-trivial integral solutions of (1).)

For $r=1$ Theorem 1 yields the polynomial solutions:

$$(10) \qquad \begin{aligned} x &= 3h^4-3h^3-3h^2+h+1, \\ y &= 3h^3-3h^2-2h+1, \\ z &= 3h^4-3h^3-3h^2+2h. \end{aligned}$$

For $r=2$ we get

$$(11) \qquad \begin{aligned} x &= (9H^2+6H)h+3H+1, \\ y &= (3H+1)h+9H^2+6H, \\ z &= (9H^2+9H+1)h, \end{aligned}$$

where $H = h^3-h^2-h$.

4. Suppose that

(12) $$x^3+y^3-z^3=x+y-z,$$

$$X=tx,\ \ Y=ty,\ \ Z=tz.$$

Then X, Y, Z will satisfy the DIOPHANTINE equation

(13) $$X^3+Y^3-Z^3=pX+pY-qZ$$

provided that

(14) $$t^2(x+y-z)=p\,(x+y)-qz.$$

But if x, y, z are chosen as in Theorem 1, then

(15) $$x+y-z=M,\ \ x+y=(h+1)M,\ \ z=hM.$$

Thus (14) and (15) show that h and t are related by the equation

(16) $$h(p-q)+p=t^2.$$

We have proved in consequence

Theorem 2. *Suppose that the integers p and q are such that the equation*

$$h(p-q)+p=t^2$$

has infinitely many integral solutions h and $t\,(\neq 0)$. Then the Diophantine equation

$$X^3+Y^3-Z^3=pX+pY-qZ$$

has infinitely many non-trivial integral solutions given by

$$X=tx(h),\ \ Y=ty(h),\ \ Z=tz(h)$$

where $x(h), y(h), z(h)$ are given by Theorem 1.

As an example, Theorem 2 provides parametric solutions for the equation

$$X^3+Y^3-Z^3=p(X+Y-Z)-Z$$

for any integer p: it is enough to take $h=p-t^2$.

But Theorem 2 fails to deal with an equation such as

$$X^3+Y^3-Z^3=10(X+Y-Z)$$

which has in fact infinitely many non-trivial integral solutions.

REFERENCES

[1] H. M. EDGAR, *Some remarks on the Diophantine equation $x^3+y^3+z^3=x+y+z$*, Proc. Amer. Math. Soc. **16** (1965), 148 — 153.

[2] A. OPPENHEIM, *On the Diophantine equation $x^3+y^3+z^3=x+y+z$*, Proc. Amer. Math. Soc. **17** (1966), 493 — 496.

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