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232. A GENERALIZATION OF A LINEAR FUNCTIONAL EQUATION

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1. D. S. Mitrinović and D. Ž. Đoković [1] have proved that the general differentiable solution of the functional equation

$$(1)\qquad (-1)^n f(x_1, x_2, \ldots, x_n) - f(x_2, x_3, \ldots, x_{n+1}) + \sum_{k=1}^{n} (-1)^{k+1} f(x_1, x_2, \ldots, x_k + x_{k+1}, \ldots, x_n, x_{n+1}) = 0$$

is given by

$$(2)\qquad f(x_1, x_2, \ldots, x_n) = (-1)^{n-1} F(x_1, x_2, \ldots, x_{n-1}) - F(x_2, x_3, \ldots, x_n) + \sum_{k=1}^{n-1} (-1)^{k+1} F(x_1, x_2, \ldots, x_k + x_{k+1}, \ldots, x_{n-1}, x_n),$$

where $F(x_1, x_2, \ldots, x_{n-1})$ is an arbitrary differentiable function.

In this paper we consider the generalization of the linear functional equation (1):

$$(3)\qquad (-1)^n F_{n+1}(x_1, x_2, \ldots, x_n) - F_{n+2}(x_2, x_3, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} F_i(x_1, x_2, \ldots, x_i + x_{i+1}, \ldots, x_n, x_{n+1}) = 0,$$

where $x_i\ (i = 1, \ldots, n+1)$ are independent variables and $F_i(x_1, x_2, \ldots, x_n)$ $(i = 1, \ldots, n+2)$ are the required functions.

We obtain the

Theorem. *All differentiable functions* $F_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, n+2)$ *satisfying the functional equation* (3) *can be written in the form*

$$F_1(x_1, x_2, \ldots, x_n) = A_{11}(x_1 + x_2, x_3, \ldots, x_n) - A_{12}(x_1, x_2 + x_3, x_4, \ldots, x_n) + \cdots + (-1)^n (A_{1\,n-1}(x_1, x_2, \ldots, x_{n-1} + x_n) + (-1)^{n+1} A_{1n}(x_1, x_2, \ldots, x_{n-1}) - C_1(x_2, x_3, \ldots, x_n),$$

Presented May 29, 1968 by D. S. Mitrinović and S. Kurepa.

$$F_2(x_1, x_2, \ldots, x_n) = A_{11}(x_1+x_2, x_3, \ldots, x_n) - (A_{12}-A_{22})(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ \cdots + (-1)^n (A_{1\,n-1} - A_{2\,n-1})(x_1, x_2, \ldots, x_{n-1}+x_n) +$$
$$+ (-1)^{n+1}(A_{1\,n} - A_{2\,n})(x_1, x_2, \ldots, x_{n-1}) - C_2(x_2, x_3, \ldots, x_n),$$

$$F_3(x_1, x_2, \ldots, x_n) = A_{12}(x_1+x_2, x_3, \ldots, x_n) - (A_{12}-A_{22})(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ \cdots + A_{13} - A_{23} + A_{33})(x_1, x_2, x_3+x_4, \ldots, x_n)$$
$$+ \cdots + (-1)^n (A_{1\,n-1} - A_{2\,n-1} + A_{3\,n-1})(x_1, x_2, \ldots, x_{n-1}+x_n)$$
$$+ (-1)^{n+1}(A_{1\,n} - A_{2\,n} + A_{3\,n})(x_1, x_2, \ldots, x_{n-1})$$
$$- C_3(x_2, x_3, \ldots, x_n),$$

$$\vdots$$

$$F_i(x_1, x_2, \ldots, x_n)$$
$$= A_{1\,i-1}(x_1+x_2, x_3, \ldots, x_n) - (A_{1\,i-1} - A_{2\,i-1})(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ (A_{1\,i-1} - A_{2\,i-1} + A_{3\,i-1})(x_1, x_2, x_3+x_4, \ldots, x_n) - \cdots$$
$$+ (-1)^i (A_{1\,i-1} - A_{2\,i-1} + \cdots + (-1)^i A_{i-1\,i-1})(x_1, x_2, \ldots, x_{i-1}+x_i, \ldots, x_n)$$
$$+ (-1)^{i+1}(A_{1\,i} - A_{2\,i} + \cdots + (-1)^{i+1} A_{ii})(x_1, x_2, \ldots, x_i+x_{i+1}, \ldots, x_n) \tag{4}$$
$$+ (-1)^{i+2}(A_{1\,i+1} - A_{2\,i+1} + \cdots + (-1)^{i+1} A_{i\,i+1})(x_1, \ldots, x_{i+1}+x_{i+2}, \ldots, x_n)$$
$$+ (-1)^n (A_{1\,n-1} - A_{2\,n-1} + \cdots + (-1)^{i+1} A_{in-1})(x_1, x_2, \ldots, x_{n-1}+x_n)$$
$$+ (-1)^{n+1}(A_{1\,n} - A_{2\,n} + \cdots + (-1)^{i+1} A_{in})(x_1, x_2, \ldots, x_{n-1})$$
$$- C_i(x_2, x_3, \ldots, x_n),$$

$$\vdots$$

$$F_n(x_1, x_2, \ldots, x_n)$$
$$= A_{1\,n-1}(x_1+x_2, x_3, \ldots, x_n) - (A_{1\,n-1} - A_{2\,n-1})(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ \cdots + (-1)^n (A_{1\,n-1} - A_{2\,n-1} + \cdots + (-1)^n A_{nn-1})(x_1, x_2, \ldots, x_{n-1}+x_n)$$
$$+ (-1)^{n+1}(A_{1\,n} - A_{2\,n} + \cdots + (-1)^{n+1} A_{nn})(x_1, x_2, \ldots, x_{n-1})$$
$$- C_n(x_2, x_3, \ldots, x_n),$$

$$F_{n+1}(x_1, x_2, \ldots, x_n)$$
$$= A_{1\,n}(x_1+x_2, x_3, \ldots, x_n) - (A_{1\,n} - A_{2\,n})(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ \cdots + (-1)^n (A_{1\,n} - A_{2\,n} + \cdots + (-1)^n A_{n-1\,n})(x_1, x_2, \ldots, x_{n-1}+x_n)$$
$$+ (-1)^{n+1}(A_{1\,n} - A_{2\,n} + \cdots + (-1)^{n+1} A_{nn})(x_1, x_2, \ldots, x_{n-1})$$
$$- C_{n+1}(x_2, x_3, \ldots, x_n),$$

$$F_{n+2}(x_1, x_2, \ldots, x_n) = C_2(x_1+x_2, x_3, \ldots, x_n) - C_3(x_1, x_2+x_3, x_4, \ldots, x_n)$$
$$+ \cdots + (-1)^n C_n(x_1, x_2, \ldots, x_{n-1}+x_n)$$
$$+ (-1)^{n+1} C_{n+1}(x_1, x_2, \ldots, x_{n-1}) - C_1(x_2, x_3, \ldots, x_n),$$

where

$$(A_{1j}-A_{2j}+\cdots+(-1)^{i+1}A_{ij})(x_1, x_2, \ldots, x_{n-1}) \stackrel{\text{def}}{=} A_{1j}(x_1, x_2, \ldots, x_{n-1})$$
$$-A_{2j}(x_1, x_2, \ldots, x_{n-1})+\cdots+(-1)^{i+1}A_{ij}(x_1, x_2, \ldots, x_{n-1}),$$
$$A_{ij}(x_1, x_2, \ldots, x_{n-1}) \quad (i, j=1, 2, \ldots, n, \ i\leqslant j),$$

and

$$C_i(x_1, x_2, \ldots, x_{n-1}) \qquad (i=1, 2, \ldots, n+1),$$

are differentiable arbitrary functions.

For the sake of simplicity we carry through the proof in the case $n=2$, 3, 4, but in a manner valid also in the general case[1].

Hereafter let us denote

$$\frac{\partial F_i(x_1, x_2, \ldots, x_n)}{\partial x_1} \stackrel{\text{def}}{=} f_i(x_1, x_2, \ldots, x_n)$$

and

$$\frac{\partial A_{ij}(x_1, x_2, \ldots, x_{n-1})}{\partial x_1} \stackrel{\text{def}}{=} a_{ij}(x_1, x_2, \ldots, x_{n-1})$$

2. In the case $n=2$ we prove that

$$\begin{aligned} F_1(x_1, x_2) &= A_{11}(x_1+x_2)-A_{12}(x_1)-C_1(x_2), \\ F_2(x_1, x_2) &= A_{11}(x_1+x_2)-(A_{12}-A_{22})(x_1)-C_2(x_2), \\ F_3(x_1, x_2) &= A_{12}(x_1+x_2)-(A_{12}-A_{22})(x_1)-C_3(x_2), \\ F_4(x_1, x_2) &= C_2(x_1+x_2)-C_3(x_1)-C_1(x_2), \end{aligned} \tag{5}$$

are the general differentiable solutions of the functional equation

$$F_3(x_1, x_2)-F_4(x_2, x_3)+F_1(x_1+x_2, x_3)-F_2(x_1, x_2+x_3)=0. \tag{6}$$

In order to obtain (5) we take derivative of (6) with respect to x_1 and we get

$$f_3(x_1, x_2)+f_1(x_1+x_2, x_3)-f_2(x_1, x_2+x_3)=0. \tag{6.1}$$

By putting $x_1=0$ and integrating (6.1) we get

$$F_1(x_2, x_3)=A_{11}(x_2+x_3)-A_{12}(x_2)-C_1(x_3), \tag{6.2}$$

where

$$A_{1i}(x)=\int_{x_0}^{x} f_{i+1}(0, t)\,dt \qquad (i=1, 2).$$

[1] L'auteur n'a démontré que le résultat énoncé relatif à l'équation fonctionnelle (3) pour les cas paticuliers $n=2$, 3, 4. Il reste donc à démontrer le résultat en question pour n quelconque.

En ce qui concerne le cas $n=2$, M. Hosszú (*On a class of functional equations*, Publ. Inst. Math. Beograd **3** (17) (1963), 53—55) a résolu l'équation correspondante n'imposant aucune hypothèse aux fonctions f_1, f_2, f_3, f_4 (Observation du Comité de rédaction).

Putting (6.2) into (6.1) we get

(6.3) $$f_3(x_1, x_2) + a_{11}(x_1 + x_2 + x_3) - a_{12}(x_1 + x_2) - f_2(x_1, x_2 + x_3) = 0.$$

Putting $x_2 = 0$ and integrating, we get

(6.4) $$F_2(x_1, x_3) = A_{11}(x_1 + x_3) - A_{12}(x_1) + A_{22}(x_1) - C_2(x_3),$$

where $A_{22}(x) = F_3(x, 0)$.

Putting (6.4) into (6.3) and integrating we find

(6.5) $$F_3(x_1, x_2) = A_{12}(x_1 + x_2) - A_{12}(x_1) + A_{22}(x_1) - C_3(x_2).$$

Putting (6.2), (6.4) and (6.5) into (6), we obtain

$$F_4(x_2, x_3) = C_2(x_2 + x_3) - C_3(x_2) - C_1(x_3).$$

Thus we have proved that (6) implies (5).

3. In the case $n = 3$ we prove that

$$F_1(x_1, x_2, x_3) = A_{11}(x_1 + x_2, x_3) - A_{12}(x_1, x_2 + x_3) + A_{13}(x_1, x_2) - C_1(x_2, x_3),$$

$$F_2(x_1, x_2, x_3) = A_{11}(x_1 + x_2, x_3) - (A_{12} - A_{22})(x_1, x_2 + x_3) + (A_{13} - A_{23})(x_1, x_2) - C_2(x_2, x_3),$$

$$F_3(x_1, x_2, x_3) = A_{12}(x_1 + x_2, x_3) - (A_{12} - A_{22})(x_1, x_2 + x_3)$$

(7) $$+ (A_{13} - A_{23} + A_{33})(x_1, x_2) - C_3(x_2, x_3),$$

$$F_4(x_1, x_2, x_3) = A_{13}(x_1 + x_2, x_3) - (A_{13} - A_{23})(x_1, x_2 + x_3) + (A_{13} - A_{23} + A_{33})(x_1, x_2) - C_4(x_2, x_3),$$

$$F_5(x_1, x_2, x_3) = C_2(x_1 + x_2, x_3) - C_3(x_1, x_2 + x_3) + C_4(x_1, x_2) - C_1(x_2, x_3)$$

are the general differentiable solutions of the functional equation

(8) $$-F_4(x_1, x_2, x_3) - F_5(x_2, x_3, x_4) + F_1(x_1 + x_2, x_3, x_4) - F_2(x_1, x_2 + x_3, x_4) + F_3(x_1, x_2, x_3 + x_4) = 0.$$

By taking derivative of (8) with respect to x_1 we get

(8.1) $$-f_4(x_1, x_2, x_3) + f_1(x_1 + x_2, x_3, x_4) - f_2(x_1, x_2 + x_3, x_4) + f_3(x_1, x_2, x_3 + x_4) = 0.$$

Putting $x_1 = 0$ and integrating, we get

(8.2) $$F_1(x_2, x_3, x_4) = A_{11}(x_2 + x_3, x_4) - A_{12}(x_2, x_3 + x_4) + A_{13}(x_2, x_3) - C_1(x_3, x_4),$$

where

$$A_{1i}(x_1, x_2) = \int_{x_0}^{x_1} f_{i+1}(0, t, x_2)\, dt \qquad (i = 1, 2, 3).$$

Put (8.2) into (8.1)

$$(8.3)\qquad -f_4(x_1, x_2, x_3)+a_{11}(x_1+x_2+x_3, x_4)-a_{12}(x_1+x_2, x_3+x_4) + a_{13}(x_3, x_4)-f_2(x_1, x_2+x_3, x_4)+f_3(x_1, x_2, x_3+x_4)=0.$$

Putting $x_2=0$ and integrating, we find

$$(8.4)\qquad F_2(x_1, x_3, x_4)=A_{11}(x_1+x_3, x_4)-(A_{12}-A_{22})(x_1, x_3+x_4) + (A_{13}-A_{33})(x_1, x_3)-C_2(x_3, x_4),$$

where $A_{2i}(x_1, x_2)=F_{i+1}(x_1, 0, x_2)$ $(i=2, 3)$.

We put (8.4) into (8.3):

$$(8.5)\qquad \begin{aligned} &-f_4(x_1, x_2, x_3)-a_{12}(x_1+x_2, x_3+x_4)+a_{13}(x_1+x_2, x_3)\\ &+a_{12}(x_1, x_2+x_3+x_4)-a_{22}(x_1, x_2+x_3+x_4)-a_{13}(x_1, x_2+x_3)\\ &+a_{23}(x_1, x_2+x_3)+f_3(x_1, x_2, x_3+x_4)=0. \end{aligned}$$

Putting $x_3=0$ and integrating, we get

$$(8.6)\qquad F_3(x_1, x_2, x_4)=A_{12}(x_1+x_2, x_4)-(A_{12}-A_{22})(x_1, x_2+x_4) + (A_{13}-A_{23}+A_{33})(x_1, x_2)-C_3(x_2, x_4)$$

where $A_{33}(x_1, x_2)=F_4(x_1, x_3, 0)-A_{13}(x_1+x_2, 0)$.

Putting (8.6) into (8.5) and integrating, we obtain

$$(8.7)\qquad F_4(x_1, x_2, x_3)=A_{13}(x_1+x_2, x_3)-(A_{13}-A_{23})(x_1, x_2+x_3) + (A_{13}-A_{23}+A_{33})(x_1, x_2)-C_4(x_2, x_3).$$

We put (8.2), (8.4), (8.6) and (8.7) into (8) and we get

$$(8.8)\qquad F_5(x_2, x_3, x_4)=C_2(x_2+x_3, x_4)-C_3(x_2, x_3+x_4)+C_4(x_2, x_3)-C_1(x_3, x_4).$$

Thus (8) implies (7).

4. In the case $n=4$ we prove that

$$(9)\qquad \begin{aligned} F_1(x_1, x_2, x_3, x_4)&=A_{11}(x_1+x_2, x_3, x_4)-A_{12}(x_1, x_2+x_3, x_4)\\ &\quad+A_{13}(x_1, x_2, x_3+x_4)-A_{14}(x_1, x_2, x_3)-C_1(x_2, x_3, x_4),\\ F_2(x_1, x_2, x_3, x_4)&=A_{11}(x_1+x_2, x_3, x_4)-(A_{12}-A_{22})(x_1, x_2+x_3, x_4)\\ &\quad+(A_{13}-A_{23})(x_1, x_2, x_3+x_4)-(A_{14}-A_{24})(x_1, x_2, x_3)\\ &\quad-C_2(x_2, x_3, x_4),\\ F_3(x_1, x_2, x_3, x_4)&=A_{12}(x_1+x_2, x_3, x_4)-(A_{12}-A_{22})(x_1, x_2+x_3, x_4)\\ &\quad+(A_{13}-A_{23}+A_{33})(x_1, x_2, x_3+x_4)\\ &\quad-(A_{14}-A_{24}+A_{34})(x_1, x_2, x_3)-C_3(x_2, x_3, x_4), \end{aligned}$$

$$F_4(x_1, x_2, x_3, x_4) = A_{13}(x_1+x_2, x_3, x_4) - (A_{13}-A_{23})(x_1, x_2+x_3, x_4)$$
$$+ (A_{13}-A_{23}+A_{33})(x_1, x_2, x_3+x_4)$$
$$-(A_{14}-A_{24}+A_{34}-A_{44})(x_1, x_2, x_3) - C_4(x_2, x_3, x_4),$$

(9) $$F_5(x_1, x_2, x_3, x_4) = A_{14}(x_1+x_2, x_3, x_4) - (A_{14}-A_{24})(x_1, x_2+x_3, x_4)$$
$$+ (A_{14}-A_{24}+A_{34})(x_1, x_2, x_3+x_4)$$
$$-(A_{14}-A_{24}+A_{34}-A_{44})(x_1, x_2, x_3) - C_5(x_2, x_3\ x_4),$$

$$F_6(x_1, x_2, x_3, x_4) = C_2(x_1+x_2, x_3, x_4) - C_3(x_1, x_2+x_3, x_4)$$
$$+ C_4(x_1, x_2, x_3+x_4) - C_5(x_1, x_2, x_3) - C_1(x_2, x_3, x_4)$$

are general differentiable solutions of the functional equation

(10) $$F_5(x_1, x_2, x_3, x_4) - F_6(x_2, x_3, x_4, x_5) + F_1(x_1+x_2, x_3, x_4, x_5)$$
$$-F_2(x_1, x_2+x_3, x_4, x_5) + F_3(x_1, x_2, x_3+x_4, x_5) - F_4(x_1, x_2, x_3, x_4+x_5) = 0.$$

In fact, differentiation of (10) with respect to x_1 leads to

(10.1) $$f_5(x_1, x_2, x_3, x_4) + f_1(x_1+x_2, x_3, x_4, x_5) - f_2(x_1, x_2+x_3, x_4, x_5)$$
$$+ f_3(x_1, x_2, x_3+x_4, x_5) - f_4(x_1, x_2, x_3, x_4+x_5) = 0.$$

Putting $x_1 = 0$ and integrating, we get

(10.2) $$F_1(x_1, x_2, x_3, x_4) = A_{11}(x_1+x_2, x_3, x_4) - A_{12}(x_1, x_2+x_3, x_4)$$
$$+ A_{13}(x_1, x_2, x_3+x_4) - A_{14}(x_1, x_2, x_3) - C_1(x_2, x_3, x_4),$$

where

$$A_{1i}(x_1, x_2, x_3) = \int_{x_0}^{x_1} f_{i+1}(0, t, x_2, x_3)\, dt \qquad (i = 1, 2, 3, 4).$$

Put (10.2) into (10.1)

(10.3) $$f_5(x_1, x_2, x_3, x_4) + a_{11}(x_1+x_2+x_3, x_4, x_5) - a_{12}(x_1+x_2, x_3+x_4, x_5)$$
$$+ a_{13}(x_1+x_2, x_3, x_4+x_5) - a_{14}(x_1+x_2, x_3, x_4) - f_2(x_1, x_2+x_3, x_4, x_5)$$
$$+ f_3(x_1, x_2, x_3+x_4, x_5) - f_4(x_1, x_2, x_3, x_4+x_5) = 0.$$

Putting $x_2 = 0$ and integrating, we find

(10.4) $$F_2(x_1, x_3, x_4, x_5) = A_{11}(x_1+x_3, x_4, x_5) - (A_{12}-A_{22})(x_1, x_3+x_4, x_5)$$
$$+ (A_{13}-A_{23})(x_1, x_3, x_4+x_5) - (A_{14}-A_{24})(x_1, x_3, x_4)$$
$$-C_2(x_3, x_4, x_5),$$

where $A_{2i}(x_1, x_2, x_3) = F_{i+1}(x_1, 0, x_2, x_3)$ $(i = 2, 3, 4)$.

Put (10.4) into (10.3)

$$(10.5)\quad f_5(x_1, x_2, x_3, x_4) - a_{12}(x_1+x_2, x_3+x_4, x_5) + a_{13}(x_1+x_2, x_3, x_4+x_5)$$
$$-a_{14}(x_1+x_2, x_3, x_4) + (a_{12}-a_{22})(x_1, x_2+x_3+x_4, x_5)$$
$$-(a_{13}-a_{23})(x_1, x_2+x_3, x_4+x_5) + (A_{14}-A_{24})(x_1, x_2+x_3, x_4)$$
$$+f_3(x_1, x_2, x_3+x_4, x_5) - f_4(x_1, x_2, x_3, x_4+x_5) = 0.$$

Putting $x_3=0$ and integrating, we get

$$(10.6)\quad F_3(x_1, x_2, x_4, x_5) = A_{12}(x_1+x_2, x_4, x_5) - (A_{12}-A_{22})(x_1, x_2+x_4, x_5)$$
$$+(A_{13}-A_{23}+A_{33})(x_1, x_2, x_4+x_5)$$
$$-(A_{14}-A_{24}+A_{34})(x_1, x_2, x_4) - C_3(x_2, x_4, x_5),$$

where $A_{3i}(x_1, x_2, x_3) = F_{i+1}(x_1, x_2, 0, x_3) - A_{1i}(x_1+x_2, 0, x_3)$ $(i=3, 4)$.

Put (10.6) into (10.5)

$$(10.7)\quad f_5(x_1, x_2, x_3, x_4) + a_{13}(x_1+x_2, x_3, x_4+x_5) - a_{14}(x_1+x_2, x_3, x_4)$$
$$-(a_{13}-a_{23})(x_1, x_2+x_3, x_4+x_5) + (A_{14}-A_{24})(x_1, x_2+x_3, x_4)$$
$$+(A_{13}-A_{23}+A_{33})(x_1, x_2, x_3, x_4+x_5) - (A_{14}-A_{24}+A_{34})(x_1, x_2, x_3+x_4)$$
$$-f_4(x_1, x_2, x_3, x_4+x_5) = 0.$$

Putting $x_4=0$ and integrating, we get

$$(10.8)\quad F_4(x_1, x_2, x_3, x_5) = A_{13}(x_1+x_2, x_3, x_5) - (A_{13}-A_{23})(x_1, x_2+x_3, x_5)$$
$$+(A_{13}-A_{23}+A_{33})(x_1, x_2, x_3+x_5)$$
$$-(A_{14}-A_{24}+A_{34}-A_{44})(x_1, x_2, x_3) - C_4(x_2, x_3, x_5),$$

where $A_{44}(x_1, x_3, x_3) = F_5(x_1, x_2, x_3, 0) - A_{14}(x_1+x_2, x_3, 0)$
$$+(A_{14}-A_{24})(x_1, x_2+x_3, 0).$$

Put (10.8) into (10.7)

$$f_5(x_1, x_2, x_3, x_4) - a_{14}(x_1+x_2, x_3, x_4) + (a_{14}-a_{24})(x_1, x_2+x_3, x_4)$$
$$-(a_{14}-a_{24}+a_{34})(x_1, x_2, x_3+x_4) + (a_{14}-a_{24}+a_{34}-a_{44})(x_1, x_2, x_3) = 0.$$

Integrating

$$(10.9)\quad F_5(x_1, x_2, x_3, x_4) = A_{14}(x_1+x_2, x_3, x_4) - (A_{14}-A_{24})(x_1, x_2+x_3, x_4)$$
$$+(A_{14}-A_{24}+A_{34})(x_1, x_2, x_3+x_4)$$
$$+(A_{14}-A_{24}+A_{34}-A_{44})(x_1, x_2, x_3) - C_5(x_2, x_3, x_4).$$

Putting (10.2), (10.4), (10.6), (10.8) and (10.9) into (10), we get

$$(10.10)\quad F_6(x_1, x_2, x_3, x_4) = C_2(x_1+x_2, x_3, x_4) - C_3(x_1, x_2+x_3, x_4) + C_4(x_1, x_2, x_3+x_4) - C_5(x_1, x_2, x_3) - C_1(x_2, x_3, x_4).$$

Thus (10) implies (9).

REFERENCE

[1] D. S. Mitrinović et D. Ž. Đoković, *Sur certaines équations fonctionnelles*, these Publications № 51—№ 54 (1961), 9—16.