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212. INEQUALITIES CONCERNING THE ELEMENTARY SYMMETRIC FUNCTIONS

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0. The elementary symmetric functions c_r of $x_1, \ldots, x_n$ are defined by

$$(x+x_1)(x+x_2)\cdots(x+x_n)=x^n+c_1x^{n-1}+c_2x^{n-2}+\cdots+c_n.$$

If $x_1, \ldots, x_n$ are real, then the following inequality holds [1]

$$c_{r-1}c_{r+1}\leqslant c_r^2 \qquad (1\leqslant r<n), \tag{1}$$

with $c_0=1$.

In what follows, we exclude the case in which equality occurs in (1).

1. Suppose that all x_r are different. If $\overline{c}_r$ denotes the r-th elementary symmetric function of $x_1, \ldots, x_{n-1}$, we have

$$c_r=\overline{c}_r+x_n\overline{c}_{r-1}. \tag{2}$$

We shall consider the difference

$$f(x_n)=c_{r-1}c_{r+1}-c_r^2 \tag{3}$$

as a function of the variable x_n. Using (2), we get

$$\begin{aligned} f(x_n)&=(\overline{c}_{r-1}+x_n\overline{c}_{r-2})(\overline{c}_{r+1}+x_n\overline{c}_r)-(\overline{c}_r+x_n\overline{c}_{r-1})^2\\ &=(\overline{c}_{r-1}\overline{c}_{r+1}-\overline{c}_r^{\,2})+(\overline{c}_{r-2}\overline{c}_{r+1}-\overline{c}_{r-1}\overline{c}_r)x_n\\ &\quad+(\overline{c}_{r-2}\overline{c}_r-\overline{c}_{r-1}^{\,2})x_n^2. \end{aligned} \tag{4}$$

By differentiation we obtain

$$f'(x_n)=(\overline{c}_{r-2}\overline{c}_{r+1}-\overline{c}_{r-1}\overline{c}_r)+2(\overline{c}_{r-2}\overline{c}_r-\overline{c}_{r-1}^{\,2})x_n, \tag{5}$$

$$f''(x_n)=2(\overline{c}_{r-2}\overline{c}_r-\overline{c}_{r-1}^{\,2}). \tag{6}$$

From (1) we conclude that $f''(x_n)<0$. The only extreme point of f is the maximum for

$$x_n = -\frac{\bar{c}_{r-2}\,\bar{c}_{r+1}-\bar{c}_{r-1}\,\bar{c}_r}{2(\bar{c}_{r-2}\,\bar{c}_r-\bar{c}_{r-1}^2)}. \tag{7}$$

The maximal value is

$$\max f(x_n)=\bar{c}_{r-1}\,\bar{c}_{r+1}-\bar{c}_r^2-\frac{1}{4}\,\frac{(\bar{c}_{r-2}\,\bar{c}_{r+1}-\bar{c}_{r-1}\,\bar{c}_r)^2}{\bar{c}_{r-2}\,\bar{c}_r-\bar{c}_{r-1}^2}.$$

Hence, we have established the inequality

$$c_{r-1}\,c_{r+1}-c_r^2\leqslant\bar{c}_{r-1}\,\bar{c}_{r+1}-\bar{c}_r^2-\frac{1}{4}\,\frac{(\bar{c}_{r-2}\,\bar{c}_{r+1}-\bar{c}_{r-1}\,\bar{c}_r)^2}{\bar{c}_{r-2}\,\bar{c}_r-\bar{c}_{r-1}^2}, \tag{8}$$

where $r<n-1$.

2. If all x_r are positive, we shall prove that

$$c_{r-2}\,c_{r+1}-c_{r-1}\,c_r<0. \tag{9}$$

From (1) we have

$$c_{r-2}\,c_r<c_{r-1}^2.$$

Multiplying both sides by c_{r+1} ($c_{r+1}>0$ since all x_r are positive), we get

$$c_{r-2}\,c_r\,c_{r+1}<c_{r-1}\,c_{r-1}\,c_{r+1}.$$

Using (1) we obtain

$$c_{r-2}\,c_r\,c_{r+1}<c_{r-1}\,c_r^2, \quad \text{i. e.,} \quad c_{r-2}\,c_{r+1}<c_{r-1}\,c_r,$$

q. e. d.

Since $x_n>0$, from (1) and (9) we conclude that $f'(x_n)<0$, i. e., $f(x_n)$ is decreasing for $x_n>0$. As $f(0)=\bar{c}_{r-1}\,\bar{c}_{r+1}-\bar{c}_r^2$ we obtain the following inequality

$$c_{r-1}\,c_{r+1}-c_r^2\leqslant\bar{c}_{r-1}\,\bar{c}_{r+1}-\bar{c}_r^2,$$

which is sharper than (8), since

$$-\frac{1}{4}\,\frac{(\bar{c}_{r-2}\,\bar{c}_{r+1}-\bar{c}_{r-1}\,\bar{c}_r)^2}{\bar{c}_{r-2}\,\bar{c}_r-\bar{c}_{r-1}^2}>0.$$

If all x_r are positive, it follows that

$$c_{r-1}\,c_{r+1}-c_r^2\leqslant\min_{1\leqslant i\leqslant n}\,({}^i c_{r-1}\,{}^i c_{r+1}-{}^i c_r^2),$$

where ${}^i c_r$ is r-th elementary symmetric function of $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$.

All inequalities remain valid if not all x_r are different.

We have not met these inequalities in the literature.

Remark 1. Using the suggestion of a referee from USA, we deduce from $c_{r-1}\,c_{r+1} \leqslant c_r^2$ $(1 \leqslant r < n)$ and (8) the following inequality:

$$(10) \qquad 4\,(c_{r-1}\,c_{r+1} - c_r^2)\,(c_{r-2}\,c_r - c_{r-1}^2) \geqslant (c_{r-2}\,c_{r+1} - c_{r-1}\,c_r)^2 \qquad (r < n-1).$$

This inequality is perhaps of some interest since all the factors are familiar expressions.

Remark 2. J. Mařík [2] gives the following result:

Let $n \geqslant 3$ be an integer number and let $a_0, a_1, \ldots, a_n$, with $a_0\,a_n \neq 0$, be real numbers such that $f(x) = \sum_{j=0}^{n} \frac{1}{j!}\,\frac{1}{(n-j)!}\,a_j\,x^{n-j}$ is the polynomial, all of whose zeros are real. Then

$$(11) \qquad 4\,(a_{j+1}^2 - a_j\,a_{j+2})\,(a_{j+2}^2 - a_{j+1}\,a_{j+3}) \geqslant (a_{j+1}\,a_{j+2} - a_j\,a_{j+3})^2,$$

for $j = 0, 1, \ldots, n-3$.

It would be of some interest to examine the connection between of the inequalities (10) and (11).

Remark 3. See also [3] and [4].

REFERENCES

[1] G. H. Hardy, J. E. Littlewood and G. Pólya: *Inequalities*, Cambridge, 1952, p. 52.

[2] J. Mařík: *Über Polynome, deren sämtliche Wurzeln reell sind*, Časopis pro pěstování matematiky, **89** (1964), 5—9.

[3] D. S. Mitrinović, *Certain inequalities for elementary symmetric functions*, these Publications, № **181—186** (1967), 17—20.

[4] D. S. Mitrinović, *Some inequalities involving elementary symmetric functions*, these Publications, № **181—196** (1967), 21—27.