PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA – SÉRIE: MAHÉMATIQUES ET PHYSIQUE

№ 206 (1967)

A GENERALIZATION OF A PROBLEM OF L. CARLITZ*

Radovan R. Janić

L. CARLITZ [1] proposed the following problem:

Let P be an interior point of a triangle ABC. Let x, y, z denote the distances from P to the vertices of ABC and let p, q, r denote the perpendiculars from P to the sides of ABC. Let α, β and γ denote the angles of ABC. Show that

$$x \sin \frac{\alpha}{2} + y \sin \frac{\beta}{2} + z \sin \frac{\gamma}{2} \geqslant p + q + r,$$

with equality only if P is the incenter of ABC.

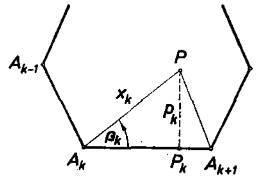
In this Note we shall prove a more general result.

Let P be an arbitrary interior point of the convex polygon $A_1A_2 \ldots A_n$. Let x_k denote the distance from P to the vertex A_k. Let p_k denote the perpendicular from P to the side A_kA_{k+1}. Let α_k be the angle of the polygon i.e., $\alpha_k = \sphericalangle A_{k-1}A_kA_{k+1}$. Then

$$\sum_{k=1}^{n} x_k \sin \frac{\alpha_k}{2} \geqslant \sum_{k=1}^{n} p_k,$$

with equality holds if and only if all sides of the polygon are tangent to a circle with center P.

Proof. Let $\beta_k = \sphericalangle PA_kA_{k+1}$. From the triangle PA_kP_k, where P_k is the feet of the perpendicular on the side A_kA_{k+1} from P (see Figure), we obtain



(1) $$p_k = x_k \sin \beta_k \qquad (k = 1, \ldots, n).$$

From the triangle $PA_{k+1}P_k$, we get

(2) $$p_k = x_{k+1} \sin (\alpha_{k+1} - \beta_{k+1}) \qquad (k = 1, \ldots, n-1),$$

(3) $$p_n = x_1 \sin (\alpha_1 - \beta_1).$$

* Presented November 1, 1967 by D. S. Mitrinović.

Adding together (1), (2) and (3), we have

$$2\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} x_k (\sin(\alpha_k - \beta_k) + \sin\beta_k),$$

i. e.,

$$2\sum_{k=1}^{n} p_k = 2\sum_{k=1}^{n} x_k \sin\frac{\alpha_k}{2}\cos\left(\frac{\alpha_k}{2} - \beta_k\right). \tag{4}$$

Using $\cos\left(\frac{\alpha_k}{2} - \alpha_k\right) \leqslant 1$, (4) gives

$$\sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} x_k \sin\frac{\alpha_k}{2}.$$

The equality holds if and only if $\cos\left(\frac{\alpha_k}{2} - \beta_k\right) = 1$, i.e., $\alpha_k = 2\beta_k$. This condition is satisfied if the sides of the polygon are tangent to a circle with center P.

REFERENCE

[1] L. Carlitz, *Aufgabe* 534, Elemente der Mathematik 21 (1966), 115.