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INEQUALITIES FOR A SIMPLEX AND AN INTERNAL POINT*

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P is an internal point of the simplex $A_0 A_1 \ldots A_n$; $x_i = PA_i$: $A_i P$ meets opposite face in $B_i : PB_i = y_i$. Inequalities are obtained between the sets x_i and y_i. (CARLITZ for the triangle; GABAI for simplex; my results (independent of GABAI's) overlap with his.)

P is also an internal point of the simplex $B_0 \ldots B_n$. Let $B_i P$ meet the opposite face in $C_i : PC_i = z_i$. Then aplication of inequalities for x_i, y_i yields inequalities for y_i, z_i and so new inequalities for x_i, y_i.

We know that

$$t_i (x_i + y_i) = y_i, \quad 0 < t_i < 1, \quad \sum t_i = 1. \tag{1}$$

Hence

$$\sum y_i = \sum t_i (x_i + y_i) \leqslant (\sum t_i) \max (x_i + y_i) = \max (x_i + y_i).$$

$$\sum y_i < \sum x_i \cdot (\text{CARLITZ for } n = 2.) \tag{2}$$

No improvement on (2) is possible. (Use CARLITZ's example for $n = 2$. Take $A_2, \ldots, A_n$ close to mid point of $A_0 A_1$; P at mid point. Then $\sum x_i - \sum y_i \to 0$ so that no inequality of type $\sum x_i \geqslant k \sum y_i$ for fixed $k > 1$ can be valid.)

Theorem. *For all positive* e_i

$$\sum x_i e_i^2 \geqslant 2 \sum (y_i y_j)^{\frac{1}{2}} e_i e_j . \tag{3}$$

Equality if the $t_i / e_i \sqrt{y_i}$ *are all equal.*

The form $\sum x_i \xi_i^2 - 2 \sum (y_i y_j)^{\frac{1}{2}} \xi_i \xi_j$ is non-negative definite.

Proof.
$$x_i = \frac{1 - t_i}{t_i} y_i = \sum \frac{t_j}{t_i} y_i \qquad (j \neq i).$$

$$\sum x_i e_i^2 = \sum \sum \left(\frac{t_j}{t_i} y_i e_i^2 + \frac{t_i}{t_j} y_j e_j^2 \right) \qquad (i \neq j)$$

$$\geqslant 2 \sum (y_i y_j)^{\frac{1}{2}} e_i e_j ,$$

etc.

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Hence in particular

(4) $$\sum \frac{x_i}{y_i} \geqslant n(n+1) \qquad (e_i = y_i^{-\frac{1}{2}}),$$

(5) $$\sum x_i \geqslant 2 \sum (y_i y_j)^{\frac{1}{2}} \qquad (e_i = 1),$$

(6) $$\sum x_i y_i \geqslant 2 \sum y_i y_j \qquad (e_i = y_i^{\frac{1}{2}}).$$

Note also that in (3) y_i can be replaced by p_i (perpendicular from P on face opposite to A_i). Hence (4), (5), (6) with p_i in place of y_i.

Since the form $\sum x_i \xi_i^2 - 2 \sum (y_i y_j)^{\frac{1}{2}} \xi_i \xi_j$ is non-negative definite (indeed in general positive definite) its principal minors are positive or zero. Thus e.g.

$$x_1 x_2 x_3 - 2y_1 y_2 y_3 - x_1 y_2 y_3 - x_2 y_3 y_1 - x_3 y_1 y_2 \geqslant 0.$$

Hence also

$$6 y_1 y_2 y_3 \leqslant \sum x_1 y_2 y_3 \quad \text{(from (3) by appropriate choice of } e_i)$$
$$\leqslant x_1 x_2 x_3 - 2 y_1 y_2 y_3$$

so that $x_1 x_2 x_3 \geqslant 8 y_1 y_2 y_3$.

Other inequalities of this nature can be found in the same way.

Inequalities for $\sum \left(\frac{x_i}{y_i}\right)^k$, $k>0$.

We have $$\frac{x_i}{y_i} = \sum_{j \neq i} \frac{t_j}{t_i} \geqslant \frac{n \left(\prod_{j \neq i} t_j\right)^{\frac{1}{n}}}{t_i} = n \left(\prod t_j\right)^{\frac{1}{n}} t_i^{-1-\frac{1}{n}};$$

$$\sum \left(\frac{x_i}{y_i}\right)^k \geqslant n^k \left(\prod t_j\right)^{\frac{k}{n}} \left(\prod t_i^{-k-\frac{k}{n}}\right)^{\frac{1}{n+1}} (n+1) = (n+1) n^k.$$

Thus for $k>0$

(7) $$\sum \left(\frac{x_i}{y_i}\right)^k \geqslant (n+1) n^k;$$

equality if and only if P is centroid of $A_0, \ldots, A_n$.

[Hence $\sum \exp \left(\frac{x_i}{y_i}\right) \geqslant (n+1) e^n$ and so on.]

Note also that from (7)

(7′) $$\sum \left(\frac{x_i}{p_i}\right)^k \geqslant (n+1) n^k \qquad (k>0)$$

where p_i is the perpendicular from P on face opposite A_i.

The simplex $B_0 B_1 \ldots B_n$, P internal. $B_i P$ meets opposite face in C_i: $PC_i = z_i$. Relations between x_i, y_i, z_i.

If $x_i' = B_iP$, $y_i' = PC_i$, then

$$x_i' = y_i, \qquad y_i' = \frac{x_i y_i}{(n-1)\,x_i + n y_i}. \tag{8}$$

Hence any statements which hold for x_i, y_i will also hold for x_i', y_i' and therefore for

y_i in place of x_i; $\dfrac{x_i y_i}{(n-1)\,x_i + n y_i}$ in place of y_i.

Thus (4) yields

$$\sum \frac{(n-1)\,x_i + n y_i}{x_i} \geqslant n\,(n+1)$$

and so

$$\sum \frac{y_i}{x_i} \geqslant 1 + \frac{1}{n} \tag{9}$$

(Stronger than the inequality for $n=2$ obtained by CARLITZ). Equality at centroid only.

[If we deal with homogeneous statements it is enough to replace x_i, y_i by $(n-1)\,x_i + n y_i$ and x_i respectively.]

Thus (3) yields

$$\sum\,[(n-1)\,x_i + n y_i]\,e_i^2 \geqslant 2 \sum\,(x_i x_j)^{\frac{1}{2}}\,e_i e_j \tag{10}$$

and in particular

$$(n-1)\sum x_i^2 + n \sum x_i y_i \geqslant 2 \sum x_i x_j. \tag{11}$$

From (7) we get, for $k>0$,

$$\sum \left(n-1+n\frac{y_i}{x_i}\right)^k \geqslant (n+1)\,n^k, \tag{12}$$

equality only at centroid.

Noteworthly also is

$$(n-1)\sum x_i + n \sum y \cdot \geqslant 2 \sum\,(x_i x_j)^{\frac{1}{2}}. \tag{13}$$

<u>*Inequalities for*</u> $\sum \left(\dfrac{y_i}{x_i}\right)^k$, $\quad (k \geqslant 1)$.

We know that

$$\left(\frac{1}{N}\sum_{i=1}^{N} a_i^r\right)^{\frac{1}{r}} \geqslant \frac{1}{N}\sum_{i=1}^{N} a_i \qquad (a_i>0,\ r \geqslant 1).$$

Hence

$$\left[\frac{1}{n+1}\sum_{i=0}^{n}\left(\frac{y_i}{x_i}\right)^k\right]^{\frac{1}{k}} \geqslant \frac{1}{n+1}\sum \frac{y_i}{x_i} \qquad (k \geqslant 1)$$

$$\geqslant \frac{1}{n}$$

by (9). Thus

$$\Sigma \left(\frac{y_i}{x_i}\right)^k \geqslant (n+1)\, n^{-k} \qquad (k \geqslant 1); \tag{14}$$

equality holds if P is centroid of simplex.

This inequality does not necessarily hold if $0<k<1$.

REFERENCES

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[2] H. GABAI, *Inequalities for simplexes,* Amer. Math. Monthly, 73 (1966), 1083 — 1087.