PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 202 (1967)

ON CERTAIN TRIANGLES INSCRIBED IN A GIVEN TRIANGLE*

O. Bottema

On the sides BC, CA, AB of a given triangle ABC we take the points A', B', C' so that $BA'=CB'=AC'=x$. The case $x \leqslant \min(a, b, c)$ has recently been considered by Ž. Živanović [1], who arrived at an inequality for the area O' of the triangle $A'B'C'$. We make here some remarks on the case that x is unrestricted, so that one or more vertices of $A'B'C'$ may lie on the extended sides of ABC.

If O is the area of ABC then

(1)
$$O' = O\left[1 - \frac{x(c-x)}{ac} - \frac{x(a-x)}{ba} - \frac{x(b-x)}{cb}\right]$$

or

(2)
$$O' = \frac{1}{4R} \cdot Q(x)$$

where

(3)
$$Q(x) \equiv (a+b+c)x^2 - (bc+ca+ab)x + abc.$$

If $Q<0$ we have $O'<0$, which means that the *orientation* of $A'B'C'$ is opposite to that of ABC; if $Q=0$ the points A', B' and C' are collinear. The discriminant D of Q reads

(4)
$$D \equiv b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2ab^2c - 2abc^2.$$

If $D<0$ the function Q is positive definite and all triangles $A'B'C'$ have the same orientation as ABC; if $D=0$ there is *one* (positive) value of x for which A', B' and C' are collinear; for $D>0$ there are *two* (positive) values of x, let us say x_1 and x_2, for which A', B' and C' are collinear and for $x_1<x<x_2$ $A'B'C'$ has an orientation which is the reverse of that of ABC.

All three cases occur: when $a=b=c$ then $D<0$; $a:b:c=1:4:4$ implies $D=0$; when $a:b:c=1:5:5$ we have $D>0$.

If the sides of a triangle are p, q, r and F is its area then

(5)
$$16F^2 = -p^4 - q^4 - r^4 + 2q^2r^2 + 2r^2p^2 + 2p^2q^2.$$

---

* Presented November 1, 1967 by D. S. Mitrinović.

Having compared this with (4) we conclude: *if a* (real, non-degenerated) *triangle with the sides* $\sqrt{bc}$, $\sqrt{ca}$ *and* $\sqrt{ab}$ *exists, then* $D<0$; *if one of these lines is greater than the sum of the other two, then* $D>0$.

A triangle will be called *special* if $D=0$. If $a \geqslant b \geqslant c$ then the triangle is special if $\sqrt{ab}=\sqrt{bc}+\sqrt{ca}$.

If an equilateral triangle $P_1P_2P_3$ is given and a, b, c are the barycentric coordinates of a point P with respect to this triangle, then P will be called the *image point* of the triangle ABC the sides of which are proportional to a, b and c. The image points of real, non-degenerated triangles are inside the triangle $P_1'P_2'P_3'$, the vertices of which are the midpoints of P_2P_3, P_3P_1, P_1P_2 (fig. 1).

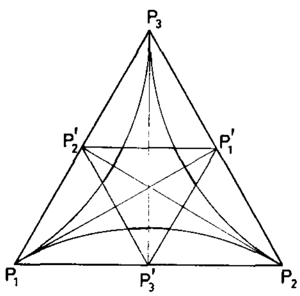

fig.1.

$D=0$ is the equation of a curve K of the fourth order and it is easily seen from (4) that it has *cusps* in the points P_1, P_2 and P_3, the cuspidal tangents being $b=c$, $c=a$ and $a=b$. Moreover, if we write

$$D \equiv (bc+ca+ab)^2-4abc\,(a+b+c)\,, \tag{6}$$

we see that K is tangent to the line at infinity $(a+b+c=0)$ at the isotropic points of the plane. In view of a theorem of Cremona [2], these characteristics are sufficient to conclude that K is Steiner's *hypocycloid.* Hence the theorem: *the locus of the image points of special triangles consists of the three arcs of Steiner's hypocycloid which lie inside the triangle* $P_1'P_2'P_3'$.

K is a rational curve. A representation in parametric form reads

$$a=t^2(1-t)^2, \quad b=(1-t)^2, \quad c=t^2, \tag{7}$$

which obviously satisfies

$$\sqrt{bc}\pm\sqrt{ca}\pm\sqrt{ab}=0\,. \tag{8}$$

The two *real* points of intersection of K and $P_1'P_2'$ $(a+b-c=0)$ are given by the roots t_1 and t_2 of $t+t^{-1}=\sqrt{2}+1$, those of K and $P_2'P_3'$ by $t_i\,(t_i-1)^{-1}$, those of K and $P_3'P_1'$ by $1-t_i$.

The midpoints of the three arcs are (1, 4, 4), (4, 1, 4) and (4, 4, 1); they correspond to the only *isosceles* special triangles.

REFERENCES

[1] Ž. Živanović, *Certaines inégalités relatives au triangle*, Ces Publications, № 181 — № 196 (1967), 69—72.

[2] G. Loria, *Spezielle algebraische und transzendente ebene Kurven* I, Leipzig und Berlin 1910, p. 161.