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A VARIATIONAL APPROACH TO THE PROBLEM OF ASYMMETRICALLY DRIVEN CYLINDRICAL ANTENNA

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SUMMARY. A variational solution for impedance of thin asymmetrically driven cylindrical antenna is derived, based on a two-term trial function for current. The impedance and current distribution parameters are expressed in terms of tabulated functions only.

1. Introduction

The problem of asymmetrically driven cylindrical antenna was treated in literature using iterative technique. [1,2] However, only the first-order solution for current distribution and impedance was attained. The aim of the present paper is to give a variational solution of that problem, using a two-term trial function for current. The variational approach was choosen because in the case of thin, symmetrically driven cylindrical antenna it yields the first-order distribution of current together with the second order values of the input impedance. [3]

2. The variational expression for impedance

Consider a dipole of radius a driven asymmetrically by a slice-generator. Let the axis of the dipole coincide with the z-axis, and let the origin coincide with the slice-generator (Fig. 1). In that case the integral equation governing the current distribution is given by

$$U\delta(z)=\frac{j\eta}{4\pi}\int_{-h_2}^{h_1} I(z')\,K(z-z')\,dz', \tag{1}$$

where $I(z')$ is the current distribution function along the antenna, and

$$\eta=(\mu_0/\varepsilon_0)^{1/2}, \tag{2}$$

$\delta(z)$ — Dirac delta function defined at $z=0$,

$$K(z-z')=k\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-jkr}}{r}, \tag{3}$$

$$r=[(z-z')^2+a^2]^{1/2}, \tag{4}$$

$$k=2\pi/\lambda=\omega(\varepsilon_0\mu_0)^{1/2}. \tag{5}$$

Multiplying (1) by $I(z)\,dz$ and integrating from $-h_2$ to h_1, and taking into account that

(6) $$U = Z\,I(0),$$

and

(7) $$\int_{-h_2}^{h_1} Z\,I(0)\,I(z)\,\delta(z)\,dz = Z\,I(0)^2$$

we obtain the fundamental variational expression for impedance

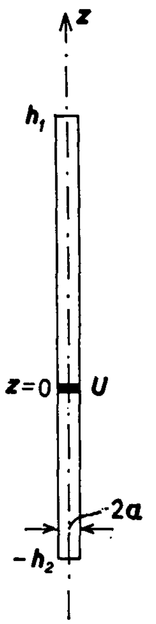

Fig. 1.

(8) $$Z\,I(0)^2 = \frac{j\eta}{4\pi}\int_{-h_2}^{h_1}\int_{-h_2}^{h_1} I(z)\,I(z')\,K(z-z')\,dz'\,dz\,.$$

Defining the normalized distribution function in respect to the current at $z=0$, i. e.

(9) $$g(z) = \frac{I(z)}{I(0)},$$

equation (8) becomes

(10) $$Z = \frac{j\eta}{4\pi}\int_{-h_2}^{h_1}\int_{-h_2}^{h_1} g(z)\,g(z')\,K(z-z')\,dz'\,dz\,.$$

Since $K(z-z') = K(z'-z)$, it can be easily shown that, as in the case of symmetrically driven dipole, Z is stationary in respect to small changes in the relative distribution function about the true distribution, i. e. that

(11) $$\delta Z = 0\,.$$

3. Expression for impedance with a two-term trial function for current

As in the case of symmetrically driven dipole, we adopt a particular two-term trial function,[3] which in the case of asymmetrically driven antenna can be put in the form

(12) $$g(z) = \begin{cases} a_1 f_1(z) + a_2 f_2(z) & \text{for } h_1 \geqslant z \geqslant 0 \\ a_3 f_3(z) + a_4 f_4(z) & \text{for } 0 \geqslant z \geqslant -h_2 \end{cases}$$

where $a_1, \dots a_4$ are complex coefficients (to be determined), and

$$f_1(z) = \sin k(h_1 - z), \tag{13}$$

$$f_2(z) = 1 - \cos k(h_1 - z), \tag{14}$$

$$f_3(z) = \sin k(h_2 + z), \tag{15}$$

$$f_4(z) = 1 - \cos k(h_2 + z). \tag{16}$$

Since at the feeding point the current must be continuous, i. e.

$$I(+0) = I(-0), \tag{17}$$

the normalized function for current must satisfy the condition

$$a_1 f_1(0) + a_2 f_2(0) = a_3 f_3(0) + a_4 f_4(0). \tag{18}$$

From the other side, since

$$g(0) = 1, \tag{19}$$

from (12) it follows that

$$a_2 = \frac{1 - a_1 f_1(0)}{f_2(0)}, \tag{20}$$

and

$$a_4 = \frac{1 - a_3 f_3(0)}{f_4(0)}. \tag{21}$$

Eliminating a_2 and a_4 from (12) using (20)-(21), we obtain

$$g(z) = \begin{cases} a_1 f_1(z) + \dfrac{1 - a_1 f_1(0)}{f_2(0)} f_2(z) & \text{for } h_1 \geqslant z \geqslant 0 \\[2ex] a_3 f_3(z) + \dfrac{1 - a_3 f_3(0)}{f_4(0)} f_4(z) & \text{for } 0 \geqslant z \geqslant -h_2 \end{cases} \tag{22}$$

Introducing (22) in (10) the input impedance Z becomes

$$Z = \frac{j\eta}{4\pi} \left\{ \int_{-h_2}^{0} \int_{-h_2}^{0} + \int_{-h_2}^{0} \int_{0}^{h_1} + \int_{0}^{h_1} \int_{-h_2}^{0} + \int_{0}^{h_1} \int_{0}^{h_1} \right\} g(z)\, g(z')\, K(z - z')\, dz'\, dz \tag{23}$$

$$= \frac{j\eta}{4\pi} \left\{ \int_{0}^{-h_2} \int_{0}^{-h_2} - \int_{0}^{-h_2} \int_{0}^{h_1} - \int_{0}^{h_1} \int_{0}^{-h_2} + \int_{0}^{h_1} \int_{0}^{h_1} \right\} g(z)\, g(z')\, K(z - z')\, dz'\, dz$$

$$= a_3^2\, w_{33} + 2\, a_3 \frac{1 - a_3 f_3(0)}{f_4(0)}\, w_{34} + \left[\frac{1 - a_3 f_3(0)}{f_4(0)} \right]^2 w_{44}$$

$$+a_3 a_1 w_{31} + a_3 \frac{1-a_1 f_1(0)}{f_2(0)} w_{32} + a_1 \frac{1-a_3 f_3(0)}{f_4(0)} w_{41}$$

$$+ \frac{1-a_3 f_3(0)}{f_4(0)} \frac{1-a_1 f_1(0)}{f_2(0)} w_{42}$$

$$+a_1 a_3 w_{13} + a_1 \frac{1-a_3 f_3(0)}{f_4(0)} w_{23} + a_3 \frac{1-a_1 f_1(0)}{f_2(0)} w_{14}$$

$$+ \frac{1-a_1 f_1(0)}{f_2(0)} \frac{1-a_3 f_3(0)}{f_4(0)} w_{24}$$

$$+a_1^2 w_{11} + 2 a_1 \frac{1-a_1 f_1(0)}{f_2(0)} w_{12} + \left[\frac{1-a_1 f_1(0)}{f_2(0)}\right]^2 w_{22}$$

where

(24) $$w_{ik} = (-1)^{m+n} \frac{j\eta}{4\pi} \int_0^{h_m} \int_0^{h_n} f_i(z) f_k(z') K(z-z')\, dz'\, dz,$$

with

(25) $$m = \begin{cases} 1 \\ 2 \end{cases} \text{ and } h_m = \begin{cases} h_1 & \text{for } i = 1,2 \\ -h_2 & \text{for } i = 3,4 \end{cases}$$

(26) $$n = \begin{cases} 1 \\ 2 \end{cases} \text{ and } h_n = \begin{cases} h_1 & \text{for } k = 1,2 \\ -h_2 & \text{for } k = 3,4 \end{cases}$$

The coefficients a_1 and a_3 entering (23) according to (11) can be determined by differentiating (23) with respect to a_1 and a_3, and requiring that

(27) $$\frac{\partial Z}{\partial a_1} = 0 \text{ and } \frac{\partial Z}{\partial a_3} = 0.$$

Hence we obtain that a_1 and a_3 are to be calculated from equations

(28) $$a_1 C + a_3 D = E,$$

(29) $$a_1 F + a_3 G = H,$$

where

(30) $$C = 2\left\{ w_{11} - 2\frac{f_1(0)}{f_2(0)} w_{12} + \left[\frac{f_1(0)}{f_2(0)}\right]^2 w_{22}\right\}$$

(31) $$D = w_{31} + w_{13} - \frac{f_1(0)}{f_2(0)}(w_{32} + w_{14}) - \frac{f_3(0)}{f_4(0)}(w_{23} + w_{41})$$

$$+ \frac{f_1(0)}{f_2(0)} \frac{f_3(0)}{f_4(0)}(w_{24} + w_{42})$$

(32)
$$E=-\frac{1}{f_4(0)}(w_{41}+w_{23})+\frac{1}{f_4(0)}\frac{f_1(0)}{f_2(0)}(w_{24}+w_{42}) -\frac{2}{f_2(0)}\left[w_{12}-\frac{f_1(0)}{f_2(0)}w_{22}\right]$$

(33)
$$F=D,$$

(34)
$$G=2\left\{w_{33}-2\frac{f_3(0)}{f_4(0)}w_{34}+\left[\frac{f_3(0)}{f_4(0)}\right]^2 w_{44}\right\}$$

(35)
$$H=-\frac{1}{f_2(0)}(w_{32}+w_{14})+\frac{1}{f_2(0)}\frac{f_3(0)}{f_4(0)}(w_{42}+w_{24}) -\frac{2}{f_4(0)}\left[w_{34}-\frac{f_3(0)}{f_4(0)}w_{44}\right].$$

Finally, if we write the non-normalized current distribution function in the form

(36)
$$I(z)=\begin{cases}U[A_1 f_1(z)+A_2 f_2(z)] & \text{for} \quad h_1 \geqslant z \geqslant 0\\ U[A_3 f_3(z)+A_4 f_4(z)] & \text{for} \quad 0 \geqslant z \geqslant -h_2\end{cases}$$

according to (9), (6) and (12) the current distribution parameters $A_1, \ldots A_4$ are given by

(37)
$$A_1=\frac{a_1}{Z}, \ldots A_4=\frac{a_4}{Z}.$$

4. Evaluation of the w-integrals

In order to transform the general integral w_{ik} given by (24), we shall utilize an identity given by Storer[3] and cited in reference 4, equation (45). By the help of that identity the integral w_{ik} can be transformed to the following form:

(38)
$$(-1)^{m+n}\frac{4\pi}{j\eta}w_{ik}=\int_0^{h_n}\left[k\int_0^{h_m}\frac{e^{-jkr}}{r}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)f_i(z)\,dz\right]f_k(z')\,dz'$$
$$+\frac{f_i(0)}{k}\left\{-f_k(0)\frac{e^{-jka}}{a}-\int_0^{h_n}\frac{e^{-jkr_0}}{r_0}f_k'(z')\,dz'\right\}$$
$$-\frac{f_i'(h_m)}{k}\int_0^{h_n}\frac{e^{-jkr_{hm}}}{r_{hm}}f_k(z')\,dz'+\frac{f_i'(0)}{k}\int_0^{h_n}\frac{e^{-jkr_0}}{r_0}f_k(z')\,dz'$$

where

(39) $$r=[a^2+(z-z')^2]^{1/2},$$

(40) $$r_0=[r]_{z=0}=(a^2+z'^2)^{1/2},$$

(41) $$r_{hm}=[r]_{z=h_m}=[a^2+(h_m-z')^2]^{1/2}.$$

Using (38), the integrals w_{ik} corresponding to the assumed Storer's current distribution, with $f_i(z)$, $i=1, 2, 3, 4$, given by (13)—(16), may be calculated in a similar manner as in reference 4. Therefore only the final formulas will be given:

(42) $$\frac{4\pi}{j\eta}w_{11}=-\sin^2 kh_1\frac{e^{-jka}}{ka}+S_a(h_1, 0),$$

(43) $$\frac{4\pi}{j\eta}w_{12}=-\sin kh_1(1-\cos kh_1)\frac{e^{-jka}}{ka}+\frac{1}{2}E_a(h_1, 0)(1-\cos kh_1),$$

(44) $$\frac{4\pi}{j\eta}w_{13}=\sin kh_1\sin kh_2\frac{e^{-jka}}{ka}$$
$$-\frac{1}{2}\{\sin k(h_1+h_2)C_a(h_2, 0)-\cos k(h_1+h_2)S_a(h_2, 0)$$
$$-\sin k(h_1+h_2)[C_a(h_1+h_2, 0)-C_a(h_1, 0)]$$
$$+\cos k(h_1+h_2)[S_a(h_1+h_2, 0)-S_a(h_1, 0)]\},$$

(45) $$\frac{4\pi}{j\eta}w_{14}=\sin kh_1(1-\cos kh_2)\frac{e^{-jka}}{ka}$$
$$-\frac{1}{2}\{-E_a(h_1+h_2, 0)+E_a(h_1, 0)+\cos kh_1 E_a(h_2, 0)$$
$$+\cos k(h_1+h_2)[C_a(h_1+h_2, 0)-C_a(h_1, 0)]$$
$$+\sin k(h_1+h_2)[S_a(h_1+h_2, 0)-S_a(h_1, 0)]$$
$$-\cos k(h_1+h_2)C_a(h_2, 0)-\sin k(h_1+h_2)S_a(h_2, 0)\}.$$

(46) $$w_{21}=w_{12},$$

(47) $$\frac{4\pi}{j\eta}w_{22}=kh_1E_a(h_1, 0)-\sin kh_1E_a(h_1, 0)+S_a(h_1, 0)$$
$$-(1-\cos kh_1)^2\frac{e^{-jka}}{ka}-2j\left(e^{-jk\sqrt{a^2+h_1^2}}-e^{-jka}\right)$$

(48)
$$\frac{4\pi}{j\eta} w_{23} = (1-\cos kh_1)\sin kh_2 \frac{e^{-jka}}{ka}$$
$$-\frac{1}{2}\{-E_a(h_1+h_2,0)+E_a(h_2,0)+\cos kh_2\, E_a(h_1,0)$$
$$+\cos k(h_1+h_2)\,[C_a(h_1+h_2,0)-C_a(h_1,0)]$$
$$+\sin k(h_1+h_2)\,[S_a(h_1+h_2,0)-S_a(h_1,0)]$$
$$-\cos k(h_1+h_2)\,C_a(h_2,0)-\sin k(h_1+h_2)\,S_a(h_2,0)\}$$

(49)
$$\frac{4\pi}{j\eta} w_{24} = (1-\cos kh_1)(1-\cos kh_2)\frac{e^{-jka}}{ka}$$
$$-j\,[e^{-jk\sqrt{a^2+(h_1+h_2)^2}}-e^{-jk\sqrt{a^2+h_1^2}}-e^{-jk\sqrt{a^2+h_2^2}}+e^{-jka}]$$
$$-\frac{1}{2}\{-kh_1\,[E_a(h_1+h_2,0)-E_a(h_1,0)]$$
$$-kh_2\,[E_a(h_1+h_2,0)-E_a(h_2,0)]$$
$$+\sin kh_2\,E_a(h_1,0)+\sin kh_1\,E_a(h_2,0)$$
$$+\sin k(h_1+h_2)\,[C_a(h_1+h_2,0)-C_a(h_1,0)]$$
$$-\cos k(h_1+h_2)\,[S_a(h_1+h_2,0)-S_a(h_1,0)]$$
$$-\sin k(h_1+h_2)\,C_a(h_2,0)+\cos k(h_1+h_2)\,S_a(h_2,0)\}$$

(50) w_{31} is obtained from w_{13}, changing h_1 to h_2, and vice versa,

(51) w_{32} is obtained from w_{23}, changing h_1 to h_2, and vice versa, except in the first term, $(1-\cos kh_1)\sin kh_2\, e^{-jka}/ka$, which remains unaltered,

(52) w_{33} is obtained from w_{11}, changing h_1 to h_2,

(53) w_{34} is obtained from w_{12}, changing h_1 to h_2,

(54) w_{41} is obtained from w_{14}, changing h_1 to h_2, and vice versa, except in the first term, $\sin kh_1(1-\cos kh_2)\, e^{-jka}/ka$, which remains unaltered,

(55) w_{42} is obtained from w_{24}, changing h_1 to h_2, and vice versa,

(56) w_{43} is obtained from w_{21}, changing h_1 to h_2,

(57) w_{44} is obtained from w_{22}, changing h_1 to h_2.

5. Conclusion

In this paper are presented concrete variational formulas for the input impedance and for current distribution parameters of a thin, asymmetrically driven cylindrical antenna, based on a two-term trial function for current. Two parts of the antenna are supposed to be of equal radii. The formulas are applicable in cases when the length of both antena parts are smaller than $3\lambda/4$.

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