PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 197 (1967)

A VARIATIONAL METHOD OF EVALUATING IMPEDANCES OF TWO COUPLED ANTENNAS OF UNEQUAL SIZES

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(Received August 15, 1967)

Summary. ***Concrete variational formulas for self and mutual impedances of two coupled antennas of unequal sizes are given, based on a two-term current distribution. The impedances are expressed in terms of tabulated functions.***

1. Introduction

The variational method for determining the impedance of a thin, cylindrical symmetrical antenna was first presented by Storer.[1,2] Some years later Levis and Tai[3] proposed an extension of Storer's variational method to the treatment of two or more coupled parallel linear antennas of unequal sizes. However, their approach was very general and no concrete final formulas were derived. Moreover, these general formulas for impedances in the case of two-term trial current included complicated complex determinants of the fourth order, whose elements are not determined but only defined in the form of integrals to be evaluated assuming a particular two-term trial current.

The aim of the present paper is to give a more concrete form to the very general approach of Levis and Tai, and to present utilisable final formulas for self- and mutual impedances of two parallel linear antennas of unequal sizes.

2. Integral equations and variational expressions for impedances

Let us consider two coupled linear antennas of unequal sizes, represented in Fig. 1.

Owing to the linearity of Maxwell's equations the currents and voltages at the antenna terminals satisfy the following relationships:

$$U_1 = Z_{11} I_1(0) + Z_{12} I_2(0), \tag{1}$$

$$U_2 = Z_{21} I_1(0) + Z_{22} I_2(0). \tag{2}$$

The integral equations governing the current distributions along the two antennas, corresponding to the feeding voltages U_1 and U_2, are given by

$$(3)\qquad U_1\,\delta(z)=\frac{j\eta}{4\pi}\int_{-h_1}^{h_1} I_1(z')\,K_{11}(z-z')\,dz' + \frac{j\eta}{4\pi}\int_{-h_2}^{h_2} I_2(z')\,K_{12}(z-z')\,dz',$$

$$(4)\qquad U_2\,\delta(z)=\frac{j\eta}{4\pi}\int_{-h_2}^{h_2} I_2(z')\,K_{22}(z-z')\,dz' + \frac{j\eta}{4\pi}\int_{-h_1}^{h_1} I_1(z')\,K_{12}(z-z')\,dz',$$

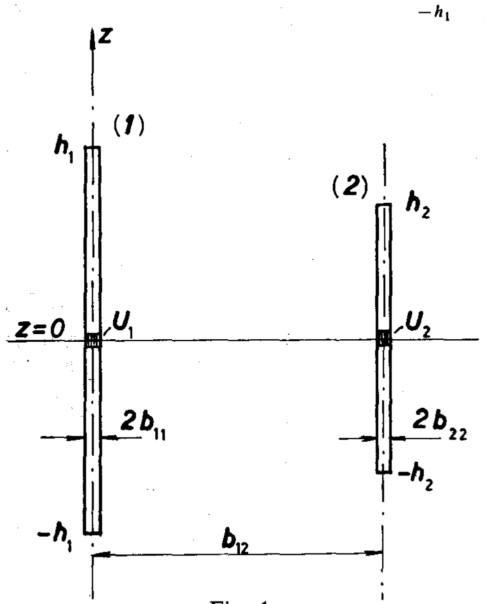

Fig. 1

where $I_1(z)$ and $I_2(z)$ are current distributions along antennas 1 and 2, and

$$\eta=(\mu_0/\varepsilon_0)^{1/2},$$

$\delta(z)$ — Dirac delta function defined at $z=0$,

$$(5)\qquad K_{11}(z-z')=K_{11}(z'-z) = k\left(1+\frac{1}{k^2}\,\frac{\partial^2}{\partial z^2}\right)\frac{e^{-jkr_{11}}}{r_{11}},$$

$$(6)\qquad r_{11}=[b_{11}{}^2+(z-z')^2]^{1/2},$$

$$(7)\qquad k=2\pi/\lambda=\omega\,(\varepsilon_0/\mu_0)^{1/2}.$$

Kernels $K_{22}(z-z')$ and $K_{12}(z-z')$ can be obtained from (5) replacing r_{11} by r_{22} viz. r_{12}, where

$$(8)\qquad r_{22}=[b_{22}{}^2+(z-z')^2]^{1/2},$$

$$(9)\qquad r_{12}=[b_{12}{}^2+(z-z')^2]^{1/2}.$$

Taking into account (1) and (2), (3) and (4) may be rewritten as follows:

$$(10)\qquad [Z_{11}\,I_1(0)+Z_{12}I_2(0)]\,\delta(z)=\frac{j\eta}{4\pi}\int_{-h_1}^{h_1} I_1(z')\,K_{11}(z-z')\,dz' + \frac{j\eta}{4\pi}\int_{-h_2}^{h_2} I_2(z')\,K_{12}(z-z')\,dz',$$

$$[Z_{12} I_1(0) + Z_{22} I_2(0)]\,\delta(z) = \frac{j\eta}{4\pi} \int_{-h_2}^{h_2} I_2(z')\, K_{22}(z-z')\, dz' + \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} I_1(z')\, K_{12}(z-z')\, dz'. \tag{11}$$

Following Levis and Tai, let us multiply (10) by $I_1(z)\,dz$ and integrate from $-h_1$ to h_1, multiply (11) by $I_2(z)\,dz$ and integrate from $-h_2$ to h_2, and add the results. Thus we obtain the equation

(12)

$$Z_{11} I_1^2(0) + 2 Z_{12} I_1(0)\, I_2(0) + Z_{22} I_2^2(0) = \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_1}^{h_1} I_1(z)\, I_1(z')\, K_{11}(z-z')\, dz'\, dz + 2\frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} I_1(z)\, I_2(z')\, K_{12}(z-z')\, dz'\, dz + \frac{j\eta}{4\pi} \int_{-h_2}^{h_2} \int_{-h_2}^{h_2} I_2(z)\, I_2(z')\, K_{22}(z-z')\, dz'\, dz,$$

which represents the basis of a variational solution for Z_{11}, Z_{12} and Z_{22}. Z_{11}, Z_{12} and Z_{22} are calculated from three equations of the type (12), corresponding to three different exitation conditions. Levis and Tai proved that impedances thus obtained have the stationary property in respect to current distributions, i. e. that

$$\delta Z_{11} = \delta Z_{12} = \delta Z_{22} = 0. \tag{13}$$

Hence, introducing reasonably good approximate current distributions in place of the true distributions, the obtained values of impedances will be approximations of a higher order to the true values of impedances.

In this paper, in order to attain the final concrete formulas for impedances, and to reduce calculating difficulties as much as possible, the following three exitation conditions will be adopted:

1° $I_1(0) \neq 0, \qquad I_2(0) = 0;$

2° $I_1(0) = 0, \qquad I_2(0) \neq 0;$ and

3° $I_1(0)/I_2(0) = -(Z_{22}/Z_{11})^{1/2}.$

In the first case, the second and third terms on the left side of (12) are zero. What concerns the right side of (12), having in mind the process of its derivation, it is easy to show that a half of the second integral and the third integral cancel, so that in exitation conditions 1° (12) becomes

$$Z_{11} I_1^2(0) = \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_1}^{h_1} I_1(z) I_1(z') K_{11}(z-z')\, dz'\, dz + \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} I_1(z) I_2(z') K_{12}(z-z')\, dz'\, dz. \tag{14}$$

In exitation conditions 2° (12) analogously becomes

$$Z_{22} I_2^2(0) = \frac{j\eta}{4\pi} \int_{-h_2}^{h_2} \int_{-h_2}^{h_2} I_2(z) I_2(z') K_{22}(z-z')\, dz'\, dz + \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} I_1(z) I_2(z') K_{12}(z-z')\, dz'\, dz. \tag{14a}$$

In exitation conditions 3° the first and third term on the left of (12) cancel, so that

$$Z_{12} I_1(0) I_2(0) = \frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_1}^{h_1} I_1(z) I_1(z') K_{11}(z-z')\, dz'\, dz + 2\frac{j\eta}{4\pi} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} I_1(z) I_2(z') K_{12}(z-z')\, dz'\, dz + \frac{j\eta}{4\pi} \int_{-h_2}^{h_2} \int_{-h_2}^{h_2} I_2(z) I_2(z') K_{22}(z-z')\, dz'\, dz. \tag{15}$$

Naturaly, current distributions corresponding to the three exitation conditions are different. However, since no possibility of confusion exists, the three sets of current distributions will not be denoted differently.

3. Variational expressions for impedances using Storer's two-term trial functions for currents

In order to make use of expressions (14)—(15) for computation of impedances, it is necessary to introduce some convenient trial functions for currents $I_1(z)$ and $I_2(z)$. It was shown by Storer[1] in the case of a single symmetrical antenna, and by Galejs[4] in the case of two short strip antennas of equal sizes, that the two-term trial function of the type

(16) $$I(z)=A\sin k(h-|z|)+B[1-\cos k(h-|z|)]$$

leeds to very satisfactory values of impedances.

The same type of trial function for currents will be used in this paper, i. e.

(17) $$I_1(z)=A_1\sin k(h_1-|z|)+A_2[1-\cos k(h_1-|z)],$$

(18) $$I_2(z)=A_3\sin k(h_2-|z|)+A_4[1-\cos k(h_2-|z|)].$$

It should be noted that complex current distribution parameters $A_1, \dots A_4$ are functions of the feeding conditions, as well as of the antenna system geometry.

In introducing those expressions in (14)—(15) it is convenient to define normalized distribution functions in respect to current $I_1(0)$ or $I_2(0)$. For example, in the case of eqn. (14) we put

(19) $$g_1(z)=\frac{I_1(z)}{I_1(0)}=a_1 f_1(z)+a_2 f_2(z),$$

(20) $$g_2(z)=\frac{I_2(z)}{I_1(0)}=a_3 f_3(z)+a_4 f_4(z),$$

where, for abbreviation,

(21) $$f_1(z)=\sin k(h_1-|z|),$$

(22) $$f_2(z)=1-\cos k(h_1-|z|),$$

(23) $$f_3(z)=\sin k(h_2-|z|),$$

(24) $$f_4(z)=1-\cos k(h_2-|z|).$$

From (19) it follows that

$$a_1 f_1(0)+a_2 f_2(0)=1,$$

wherefrom

(25) $$a_2=\frac{1-a_1 f_1(0)}{f_2(0)}.$$

Taking into account feeding conditions 1°, from (22) it follows that

(26) $$a_4=-a_3\frac{f_3(0)}{f_4(0)}.$$

Hence, for the particular feeding conditions 1°, the normalized current distribution functions (19) and (20) become

(27) $$g_1(z)=a_1 f_1(z)+\frac{1-a_1 f_1(0)}{f_2(0)}f_2(z),$$

(28) $$g_2(z)=a_3 f_3(z)-a_3\frac{f_3(0)}{f_4(0)}f_4(z).$$

Introducing (27) and (28) in (14) we obtain

$$Z_{11}=a_1^2 w_{11}+2a_1\frac{1-a_1 f_1(0)}{f_2(0)}w_{12}+\left[\frac{1-a_1 f_1(0)}{f_2(0)}\right]^2 w_{22}$$
$$+a_1 a_3 w_{13}-a_1 a_3\frac{f_3(0)}{f_4(0)}w_{14}+a_3\frac{1-a_1 f_1(0)}{f_2(0)}w_{23}$$
$$-a_3\frac{f_3(0)}{f_4(0)}\,\frac{1-a_1 f_1(0)}{f_2(0)}w_{24} \tag{29}$$

where w_{ik} stand for the following integrals

$$w_{ik}=\frac{j\eta}{4\pi}\int_{-hm}^{hm}\int_{-hn}^{hn} f_i(z)\,f_k(z')\,K_{mn}(z-z')\,dz'\,dz, \tag{30}$$

with

$$m=\begin{cases}1 & \text{for } i=1,2\\ 2 & \text{for } i=3,4\end{cases} \tag{31}$$

$$n=\begin{cases}1 & \text{for } k=1,2\\ 2 & \text{for } k=3,4\end{cases}. \tag{32}$$

In an analogous manner

$$Z_{22}=a_3^2 w_{33}+2a_3\frac{1-a_3 f_3(0)}{f_4(0)}w_{34}+\left[\frac{1-a_3 f_3(0)}{f_4(0)}\right]^2 w_{44}$$
$$+a_3 a_1 w_{31}-a_3 a_1\frac{f_1(0)}{f_2(0)}w_{41}+a_1\frac{1-a_3 f_3(0)}{f_4(0)}w_{32}$$
$$-a_1\frac{f_1(0)}{f_2(0)}\,\frac{1-a_3 f_3(0)}{f_4(0)}w_{42}, \tag{33}$$

where coefficients a_3 and a_1 now represent parameters A_3 and A_1 normalized in respect to $I_2(0)$.

In the case of exitation conditions 3°, (15) yields

$$Z_{12}=\frac{j\eta}{4\pi}\,\frac{I_1(0)}{I_2(0)}\int_{-h_1}^{h_1}\int_{-h_1}^{h_1} g_1(z)\,g_1(z')\,K_{11}(z-z')\,dz'\,dz$$
$$+2\frac{j\eta}{4\pi}\int_{-h_1}^{h_1}\int_{-h_2}^{h_2} g_1(z)\,g_2(z')\,K_{12}(z-z')\,dz'\,dz$$
$$+\frac{j\eta}{4\pi}\,\frac{I_2(0)}{I_1(0)}\int_{-h_2}^{h_2}\int_{-h_2}^{h_2} g_2(z)\,g_2(z')\,K_{22}(z-z')\,dz'\,dz \tag{34}$$

where again

$$g_1(z) = \frac{I_1(z)}{I_1(0)} = a_1 f_1(z) + a_2 f_2(z), \tag{35}$$

but

$$g_2(z) = \frac{I_2(z)}{I_2(0)} = a_3 f_3(z) + a_4 f_4(z). \tag{36}$$

From (35) and (36) it follows that

$$a_2 = \frac{1 - a_1 f_1(0)}{f_2(0)}, \tag{37}$$

$$a_4 = \frac{1 - a_3 f_3(0)}{f_4(0)}. \tag{38}$$

Putting

$$I_1(0)/I_2(0) = -(Z_{22}/Z_{11})^{1/2} = p,$$

and introducing (35)—(38) in (34), we obtain

$$\begin{aligned} Z_{12} = p &\left\{ a_1^2 w_{11} + 2a_1 \frac{1 - a_1 f_1(0)}{f_2(0)} w_{12} + \left[\frac{1 - a_1 f_1(0)}{f_2(0)}\right]^2 w_{22} \right\} \\ &+ 2\left\{ a_1 a_3 w_{13} + a_1 \frac{1 - a_3 f_3(0)}{f_4(0)} w_{14} + a_3 \frac{1 - a_1 f_1(0)}{f_2(0)} w_{23} \right. \\ &\left. + \frac{1 - a_1 f_1(0)}{f_2(0)} \frac{1 - a_3 f_3(0)}{f_4(0)} w_{24} \right\} \\ &+ \frac{1}{p} \left\{ a_3^2 w_{33} + 2a_3 \frac{1 - a_3 f_3(0)}{f_4(0)} w_{34} + \left[\frac{1 - a_3 f_3(0)}{f_4(0)}\right]^2 w_{44} \right\}. \end{aligned} \tag{39}$$

4. Evaluation of coefficients a_1 and a_3 corresponding to the three exitation conditions

4.1. Evaluation of a_1 and a_3 entering Z_{11}. Because of the stationary character of (29), as shown in (13), we shall determine the coefficients a_1 and a_3 in (29) by differentiating (29) with respect to a_1 and a_3, and requiring that

$$\frac{\partial Z_{11}}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial Z_{11}}{\partial a_3} = 0.$$

Hence we obtain

$$a_1 = \frac{\dfrac{1}{f_2(0)}\left[\dfrac{f_3(0)}{f_4(0)} w_{24} - w_{23}\right]}{w_{13} - \dfrac{f_3(0)}{f_4(0)} w_{14} - \dfrac{f_1(0)}{f_2(0)} w_{23} + \dfrac{f_1(0) f_3(0)}{f_2(0) f_4(0)} w_{24}} \tag{40a}$$

(40b) $$a_3 = \frac{\frac{2}{f_2(0)}\left[\frac{f_1(0)}{f_2(0)} w_{22} - w_{12}\right] - a_1\left\{2\, w_{11} - 4\frac{f_1(0)}{f_2(0)} w_{12} + 2\left[\frac{f_1(0)}{f_2(0)}\right]^2 w_{22}\right\}}{w_{13} - \frac{f_3(0)}{f_4(0)} w_{14} - \frac{f_1(0)}{f_2(0)} w_{23} + \frac{f_1(0) f_3(0)}{f_2(0) f_4(0)} w_{24}}$$

where

(41a) $$f_1(0) = \sin kh_1,$$

(41b) $$f_2(0) = 1 - \cos kh_1,$$

(41c) $$f_3(0) = \sin kh_2,$$

(41d) $$f_4(0) = 1 - \cos kh_2.$$

4.2. Evaluation of a_1 and a_3 entering Z_{22}. Requiring that Z_{22} given by (33) satisfies the conditions

$$\frac{\partial Z_{22}}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial Z_{22}}{\partial a_3} = 0,$$

we obtain

(42a) $$a_3 = \frac{\frac{1}{f_4(0)}\left[\frac{f_1(0)}{f_2(0)} w_{42} - w_{32}\right]}{w_{31} - \frac{f_1(0)}{f_2(0)} w_{41} - \frac{f_3(0)}{f_4(0)} w_{32} + \frac{f_1(0) f_3(0)}{f_2(0) f_4(0)} w_{42}},$$

(42b) $$a_1 = \frac{\frac{2}{f_4(0)}\left[\frac{f_3(0)}{f_4(0)} w_{44} - w_{34}\right] - a_3\left\{2\, w_{33} - 4\frac{f_3(0)}{f_4(0)} w_{34} + 2\left[\frac{f_3(0)}{f_4(0)}\right]^2 w_{44}\right\}}{w_{31} - \frac{f_1(0)}{f_2(0)} w_{41} - \frac{f_3(0)}{f_4(0)} w_{32} + \frac{f_1(0) f_3(0)}{f_2(0) f_4(0)} w_{42}}$$

$f_1(0), \ldots f_4(0)$ being given by (41a)—(41d).

4.3. Evaluation of a_1 and a_3 entering Z_{12}. Requiring that Z_{12} given by (34) satisfies the conditions

$$\frac{\partial Z_{12}}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial Z_{12}}{\partial a_3} = 0,$$

we obtain that a_1 and a_3 corresponding to these exitation conditions are to be evaluated from equations

(43) $$a_1\left\{p\left[w_{11} - 2\frac{f_1(0)}{f_2(0)} w_{12} + \frac{f_1^2(0)}{f_2^2(0)} w_{22}\right]\right\} + a_3\left\{w_{13} - \frac{f_3(0)}{f_4(0)} w_{14} - \frac{f_1(0)}{f_2(0)} w_{23} + \frac{f_1(0) f_3(0)}{f_2(0) f_4(0)} w_{24}\right\}$$
$$= p\frac{f_1(0)}{f_2^2(0)} w_{22} - \frac{p}{f_2(0)} w_{12} - \frac{1}{f_4(0)} w_{14} + \frac{f_1(0)}{f_2(0) f_4(0)} w_{24},$$

and

(44)
$$a_1\left\{w_{13}-\frac{f_3(0)}{f_4(0)}w_{14}-\frac{f_1(0)}{f_2(0)}w_{23}+\frac{f_1(0)f_3(0)}{f_2(0)f_4(0)}w_{24}\right\}$$
$$+a_3\left\{\frac{1}{p}\left[w_{33}-2\frac{f_3(0)}{f_4(0)}w_{34}+\frac{f_3^2(0)}{f_4^2(0)}w_{44}\right]\right\}$$
$$=\frac{1}{p}\frac{f_3(0)}{f_4^2(0)}w_{44}-\frac{1}{pf_4(0)}w_{34}-\frac{1}{f_2(0)}w_{23}+\frac{f_3(0)}{f_2(0)f_4(0)}w_{24},$$

where $f_1(0), \dots, f_4(0)$ are given by (41 a)—(41 d), and $p=-(Z_{22}/Z_{11})^{1/2}$.

Hence, for final determination of impedances Z_{11}, Z_{22} and Z_{12} according to formulas (29) with (40 a)—(40 b), (33) with (42 a)—(42 b), and (39) with (43)—(44), respectively, it remains only to evaluate the integrals w_{ik}.

5. Evalution of integrals w_{ik}

For evaluation of the w-integrals we shall make use of the following identity:[1] Let

$$r=[b^2+(z-z')^2]^{1/2},$$

and suppose that

$$g(h)=0$$

(corresponding to the condition that current at the antenna end vanishes), then

(45)
$$k\int_0^h g(z)\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-jkr}}{r}dz$$
$$=k\int_0^h \frac{e^{-jkr}}{r}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)g(z)\,dz$$
$$+\frac{1}{k}\left[g(0)\frac{\partial}{\partial z'}\frac{e^{-jkr_0}}{r_0}-\frac{e^{-jkr_h}}{r_h}g'(h)+\frac{e^{-jkr_0}}{r_0}g'(0)\right],$$

where

(46)
$$r_0=(b^2+z'^2)^{1/2},$$

(47)
$$r_h=[b^2+(h-z')^2]^{1/2}.$$

By the help of the above identity, and having in mind that $f_i(z)$ are symmetric functions, the general integral (30) can be transformed as follows:

$$\frac{4\pi}{j\eta}w_{ik}=\int_{-h_m}^{h_m}\int_{-h_n}^{h_n} f_i(z)f_k(z')K_{mn}(z-z')\,dz'\,dz$$

$$= k \int_{-h_m}^{h_m} \int_{-h_n}^{h_n} f_i(z) f_k(z') \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2}\right) \frac{e^{-jkr_{mn}}}{r_{mn}} dz' \, dz$$

$$= 2 \int_{-h_n}^{h_n} \left[k \int_{0}^{h_m} f_i(z) \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2}\right) \frac{e^{-jkr_{mn}}}{r_{mn}} dz \right] f_k(z') \, dz'$$

$$= 2 \int_{-h_n}^{h_n} \left[k \int_{0}^{h_m} \frac{e^{-jkr_{mn}}}{r_{mn}} \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2}\right) f_i(z) \, dz \right] f_k(z') \, dz'$$

$$+ 2 \int_{-h_n}^{h_n} \left[\frac{1}{k} f_i(0) \frac{\partial}{\partial z'} \frac{e^{-jkr_{omn}}}{r_{omn}} \right] f_k(z') \, dz'$$

$$- 2 \int_{-h_n}^{h_n} \left[\frac{1}{k} f_i'(h_m) \frac{e^{-jkr_{hmn}}}{r_{hmn}} \right] f_k(z') \, dz'$$

$$+ 2 \int_{-h_n}^{h_n} \left[\frac{1}{k} f_i'(0) \frac{e^{-jkr_{omn}}}{r_{omn}} \right] f_k(z') \, dz',$$

where r_{omn} and r_{hmn} are given by

(48) $r_{omn} = [r_{mn}]_{z=0}$

(49) $r_{hmn} = [r_{mn}]_{z=h_m}$ $\quad \Big\} \quad m = 1, 2, \quad n = 1, 2.$

Because of symmetry the integral involving $f_i(0)$ wanishes. Therefore, changing variable in the integral involving $f_i'(h_m)$, we obtain

(50)
$$\frac{4\pi}{j\eta} w_{ik} = 2 \int_{-h_n}^{h_n} \left[k \int_{0}^{h_m} \frac{e^{-jkr_{mn}}}{r_{mn}} \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2}\right) f_i(z) \, dz \right] f_k(z') \, dz$$

$$- 2 \frac{f_i'(h_m)}{k} \int_{h_m - h_n}^{h_m + h_n} f_k(h_m - z') \frac{e^{-jkr_{omn}}}{r_{omn}} dz'$$

$$+ 4 \frac{f_i'(0)}{k} \int_{0}^{h_n} f_k(z') \frac{e^{-jkr_{omn}}}{r_{omn}} dz'.$$

Let us now calculate integrals w_{ik} corresponding to the assumed Storer's current distribution, where $f_i(z)$, $i=1, 2, 3, 4$, are given by (21)—(24). Since cases $m=n$ can easily be obtained from cases $m \neq n$, only the latter will be evaluated.

These four w_{ik} will be evaluated in turn.

5.1. Evaluation of $w_{13}(=w_{31})$. In the case $i=1$, $k=3$, according to (31) and (32), $m=1$ and $n=2$. From (21) and (23) it follows that

(51) $$f_i(z)=f_1(z)=\sin k(h_1-|z|),$$

(52) $$f_k(z')=f_3(z')=\sin k(h_2-|z'|),$$

so that, for $z>0$,

$$\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)f_i(z)=0.$$

Hence, from (50),

(53) $$\frac{4\pi}{j\eta}w_{13}=2\int\limits_{h_1-h_2}^{h_1+h_2}\sin k(h_2-|h_1-z'|)\frac{e^{-jkr_{012}}}{r_{012}}dz'$$

$$-4\cos kh_1\int\limits_0^{h_2}\sin k(h_2-z')\frac{e^{-jkr_{012}}}{r_{012}}dz' \equiv J_1-J_2.$$

In what follows it will always be assumed that $h_1>h_2$.

Let us calculate first integral J_1. Since for $z'<h_1$

$$\sin k(h_2-|h_1-z'|)=\sin k(h_2-h_1+z')$$

and for $z'>h_1$

$$\sin k(h_2-|h_1-z'|)=\sin k(h_2+h_1-z'),$$

integration of J_1 will be performed in ranges $(h_1-h_2)\leqslant z'\leqslant h_1$ and $h_1\leqslant z'\leqslant \leqslant (h_1+h_2)$:

$$J_1=2\int\limits_{h_1-h_2}^{h_1}\sin k(h_2-h_1+z')\frac{e^{-jkr_{012}}}{r_{012}}dz'$$

$$+2\int\limits_{h_1}^{h_1+h_2}\sin k(h_2+h_1-z')\frac{e^{-jkr_{012}}}{r_{012}}dz'.$$

Introducing a new variable, $t=kz'$, and putting

(54) $$k\,r_{012}=R_{012}=[t^2+(kb_{12})^2]^{1/2},$$

J_1 becomes

(55)
$$J_1 = -2\sin k(h_1-h_2)\int\limits_{k(h_1-h_2)}^{kh_1}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$
$$+2\cos k(h_1-h_2)\int\limits_{k(h_1-h_2)}^{kh_1}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$
$$+2\sin k(h_1+h_2)\int\limits_{kh_1}^{k(h_1+h_2)}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$
$$-2\cos k(h_1+h_2)\int\limits_{kh_1}^{k(h_1+h_2)}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt.$$

Let us denote, as usually, integrals on the right side of (55) in the following manner[5]

(56)
$$C_b(h,0)=2\int\limits_0^{kh}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt,$$

(57)
$$S_b(h,0)=2\int\limits_0^{kh}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt.$$

$C_b(h,0)$ and $S_b(h,0)$ may be expressed in terms of tabulated generalized sine- and cosine-integral functions[6] and inverse hyperbolic sines[7] as follows (reference 5, p. 97):

(58) $C_b(h,0)=2\,[\mathrm{sh}^{-1}(h/b)-C(kb,kh)-Cc(kb,kh)]-j\,2\,Sc(kb,kh)$

(59) $S_b(h,0)=2\,Cs(kb,kh)-j\,2\,Ss(kb,kh)$.

Integral J_2 from (53) can be reduced to the form

(60)
$$J_2=4\cos kh_1\sin kh_2\int\limits_0^{kh_2}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$
$$-4\cos kh_1\cos kh_2\int\limits_0^{kh_2}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt,$$

in which again appear integrals (56) and (57).

Therefore, according to (55)—(57) and (60), (53) becomes

$$\frac{4\pi}{j\eta}w_{13} = -\sin k(h_1-h_2)[C_b(h_1,0)-C_b(h_1-h_2,0)] + \cos k(h_1-h_2)[S_b(h_1,0)-S_b(h_1-h_2,0)] + \sin k(h_1+h_2)[C_b(h_1+h_2,0)-C_b(h_1\ 0)] - \cos k(h_1+h_2)[S_b(h_1+h_2,0)-S_b(h_1,0)] - 2\cos kh_1 \sin kh_2\, C_b(h_2,0) + 2\cos kh_1 \cos kh_2\, S_b(h_2,0). \tag{61}$$

5.2. Evaluation of w_{14} $(=w_{41})$. In this case, according to (31) and (32), again $m=1$ and $n=2$, and from (21) and (23) now

$$f_i(z) = f_1(z) = \sin k(h_1-|z'|), \tag{62}$$

$$f_k(z') = f_4(z') = 1-\cos k(h_2-|z'|), \tag{63}$$

so that, for $z>0$,

$$\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)f_i(z) = 0.$$

Therefore, from (50),

$$\frac{4\pi}{j\eta}w_{14} = 2\int\limits_{h_1-h_2}^{h_1+h_2}\{1-\cos k[h_2-(h_1-|z'|)]\}\frac{e^{-jkr_{012}}}{r_{012}}dz' - 4\cos kh_1\int\limits_0^{h_2}[1-\cos k(h_2-z')]\frac{e^{-jkr_{012}}}{r_{012}}dz'. \tag{64}$$

In a quite similar manner as in the case of w_{13}, (64) may be reduced to

$$\frac{4\pi}{j\eta}w_{14} = 2\int\limits_{k(h_1-h_2)}^{k(h_1+h_2)}\frac{e^{-jR_{012}}}{R_{012}}dt - 4\cos kh_1\int\limits_0^{kh_2}\frac{e^{-jR_{012}}}{R_{012}}dt - 2\cos k(h_1-h_2)\int\limits_{k(h_1-h_2)}^{kh_1}\cos t\frac{e^{-jR_{012}}}{R_{012}}dt - 2\sin k(h_1-h_2)\int\limits_{k(h_1-h_2)}^{kh_1}\sin t\frac{e^{-jR_{012}}}{R_{012}}dt \tag{65}$$

$$-2\cos k(h_1+h_2)\int\limits_{kh_1}^{k(h_1+h_2)}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$

$$-2\sin k(h_1+h_2)\int\limits_{kh_1}^{k(h_1+h_2)}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$

$$+4\cos kh_1\cos kh_2\int\limits_{0}^{kh_2}\cos t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$

$$+4\cos kh_1\sin kh_2\int\limits_{0}^{kh_2}\sin t\,\frac{e^{-jR_{012}}}{R_{012}}\,dt$$

In (65) only the first two integrals are of a new type, usually denoted *as*[5]

(66) $$E_b(h,\,0)=2\int\limits_0^{kh}\frac{e^{-jR_{012}}}{R_{012}}\,dt.$$

$E_b(h,\,0)$ may also be expressed in terms of tabulated generalized sine- and cosine-integral functions[6] and inverse hyperbolic sines[7] as follows (reference 5, p. 97):

(67) $$E_b(h,\,0)=2\,[\mathrm{sh}^{-1}(h/b)-C(kb,\,kh)]-j\,2\,S(kb,\,kh).$$

Thus (65) can be written in the form

(68) $$\frac{4\pi}{j\eta}w_{14}=E_b(h_1+h_2,\,0)-E_b(h_1-h_2,\,0)-2\cos kh_1\,E_b(h_2,\,0)$$

$$-\cos k(h_1-h_2)\,[C_b(h_1,\,0)-C_b(h_1-h_2,\,0)]$$

$$-\sin k(h_1-h_2)\,[S_b(h_1,\,0)-S_b(h_1-h_2,\,0)]$$

$$-\cos k(h_1+h_2)\,[C_b(h_1+h_2)-C_b(h_1,\,0)]$$

$$-\sin k(h_1+h_2)\,[S_b(h_1+h_2)-S_b(h_1,\,0)]$$

$$+2\cos kh_1\cos kh_2\,C_b(h_2,\,0)$$

$$+2\cos kh_1\sin kh_2\,S_b(h_2,\,0).$$

5.3. Evaluation of $w_{23}\,(=w_{32})$. In this case again $m=1$ and $m=2$, and

(69) $$f_i(z)=f_2(z)=1-\cos k(h_1-|z|),$$

(70) $$f_k(z')=f_3(z')=\sin k(h_2-|z'|),$$

so that, for $z>0$,

$$\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)f_i(z)=1.$$

Hence, from (50),

$$\frac{4\pi}{j\eta}w_{23}=2\int_{-h_2}^{h_2}\left[k\int_0^{h_1}\frac{e^{-jkr_{12}}}{r_{12}}dz\right]\sin k(h_2-|z'|)\,dz' \tag{71}$$

$$-4\sin kh_1\int_0^{h_2}\sin k(h_2-|z'|)\frac{e^{-jkr_{012}}}{r_{012}}dz.$$

The second integral on the right in (71) can be readily expressed in terms of C_b and S_b functions. The first, double integral is of a new type, but can again be reduced to C_b, S_b and E_b functions, by changing variable in the integral in braces, and by partial integration, as follows:

$$2\int_{-h_2}^{h_2}\left[k\int_0^{h_1}\frac{e^{-jkr_{12}}}{r_{12}}dz\right]\sin k(h_2-|z'|)\,dz' \tag{72}$$

$$=2\int_{-h_2}^{0}\left[\int_{-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\right]k\sin k(h_2+z')\,dz'$$

$$+2\int_0^{h_2}\left[\int_{-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\right]k\sin k(h_2-z')\,dz'$$

$$=2\int_0^{h_2}\left[\int_{z'}^{h_1+z'}\frac{e^{-jkr_{012}}}{r_{012}}dz+\int_{-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\right]k\sin k(h_2-z')\,dz'$$

$$=2\int_0^{h_2}\left[\int_{-h_1-z'}^{-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz+\int_{-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\right]k\sin k(h_2-z')\,dz'$$

$$=2\int_0^{h_2}\left[\int_{-h_1-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\right]k\sin k(h_2-z')\,dz'$$

$$=2\cos k(h_2-z')\int_{-h_1-z'}^{h_1-z'}\frac{e^{-jkr_{012}}}{r_{012}}dz\,\Bigg|_{z'=0}^{z'=h_2}$$

$$-2\int_0^{h_2}\left[-\frac{e^{-jkr_{h12}}}{r_{h12}}+\frac{e^{-jkr_{-h12}}}{r_{-h12}}\right]\cos k\,(h_2-z')\,dz'$$

$$=2\int_{-h_1-h_2}^{h_1-h_2}\frac{e^{-jkr_{012}}}{r_{012}}\,dz-2\cos kh_2\int_{-h_1}^{h_1}\frac{e^{-jkr_{012}}}{r_{012}}\,dz$$

$$+2\int_0^{h_2}\frac{e^{-jkr_{h12}}}{r_{h12}}\cos k\,(h_2-z')\,dz'$$

$$-2\int_0^{h_2}\frac{e^{-jkr_{-h12}}}{r_{-h12}}\cos k\,(h_2-z')\,dz',$$

where

(73) $$r_{h12}=[b_{12}{}^2+(h_1-z')^2]^{1/2},$$

(74) $$r_{-h12}=[b_{12}{}^2+(-h_1-z')^2]^{1/2}=[b_{12}{}^2+(h_1+z')^2]^{1/2}.$$

By changing appropriately the variable in the third and fourth integrals on the right of (72), so that instead of r_{h12} and r_{-h12} appear $r_{012}=(b_{12}{}^2+z'^2)^{1/2}$, those two integrals reduce to the types (56) ayd (57). Hence (71) becomes

(75) $$\frac{4\pi}{j\eta}w_{23}=E_b(h_1-h_2,\,0)+E_b(h_1+h_2,\,0)-2\cos kh_2\,E_b(h_1,\,0)$$
$$+\cos k\,(h_1-h_2)\,[C_b(h_1,\,0)-C_b(h_1-h_2,\,0)]$$
$$+\sin k\,(h_1-h_2)\,[S_b(h_1,\,0)-S_b(h_1-h_2,\,0)]$$
$$-\cos k\,(h_1+h_2)\,[C_b(h_1+h_2,\,0)-C_b(h_1,\,0)]$$
$$-\sin k\,(h_1+h_2)\,[S_b(h_1+h_2,\,0)-S_b(h_1,\,0)]$$
$$-2\sin kh_1\sin kh_2\,C_b(h_2,\,0)$$
$$+2\sin kh_1\cos kh_2\,S_b(h_2,\,0).$$

5.4. Evaluation of $w_{24}(=w_{42})$. Again $m=1$ and $n=2$, and

(76) $$f_i(z)=f_2(z)=1-\cos k\,(h_1-|\,z'\,|)\,,$$

(77) $$f_k(z')=f_4(z')=1-\cos k\,(h_2-|\,z'\,|)\,,$$

so that, for $z>0$,

$$\left(1+\frac{1}{k^2}\,\frac{\partial^2}{\partial z^2}\right)f_i(z)=1\,.$$

Therefore, from (50),

$$\frac{4\pi}{j\eta} w_{24} = 2 \int_{-h_2}^{h_2} \left[k \int_0^{h_1} \frac{e^{-jkr_{12}}}{r_{12}} dz \right] [1 - \cos k (h_2 - |z'|)] dz' - 2 \sin kh_1 \int_0^{h_2} [1 - \cos k (h_2 - |z'|)] \frac{e^{-jkr_{012}}}{r_{012}} dz'. \tag{78}$$

Transforming the double integral on the right in the similar manner as in the case of w_{23}, it may de shown that (78) reduces to

$$\begin{aligned} \frac{4\pi}{j\eta} w_{24} = {} & k (h_1 + h_2) [E_b (h_1 + h_2, 0) - E_b (h_1 - h_2, 0)] \\ & - 2 \sin kh_2 E_b (h_1, 0) - 2j [e^{-j \sqrt{(kb_{12})^2 + k^2 (h_1 + h_2)^2}} \\ & - e^{-j \sqrt{(kb_{12})^2 + k^2 (h_1 - h_2)^2}}] \\ & - \sin k (h_1 - h_2) [C_b (h_1, 0) - C_b (h_1 - h_2, 0)] \\ & + \cos k (h_1 - h_2) [S_b (h_1, 0) - S_b (h_1 - h_2, 0)] \\ & - \sin k (h_1 + h_2) [C_b (h_1 + h_2, 0) - C_b (h_1, 0)] \\ & + \cos k (h_1 + h_2) [S_b (h_1 + h_2, 0) - S_b (h_1, 0)] \\ & - 2 \sin kh_1 E_b (h_2, 0) + 2 \sin kh_1 \cos kh_2 C_b (h_2, 0) \\ & + 2 \sin kh_1 \sin kh_2 S_b (h_2, 0). \end{aligned} \tag{79}$$

5.5. Evaluation of w_{ik} for $m=n$. According to (31) and (32). to $m=n$ correspond the following six w_{ik}: w_{11}, w_{12} $(=w_{21})$, w_{22} and w_{33}, w_{34} $(=w_{43})$, w_{44}. It is evident that w_{11} and w_{33} are obtained from (61), replacing h_2 by h_1 and b_{12} by b_{11}, viz. replacing h_1 by h_2 and b_{12} by b_{22}, respectively. Also, w_{12} and w_{34} are obtained from (68) or (75), replacing h_2 by h_1 and b_{12} by b_{11}, viz. replacing h_1 by h_2 and b_{12} by b_{22}, respectively. Finally, w_{22} and w_{44} are obtained from (79), replacing h_2 by h_1 and b_{12} by b_{11}, viz. h_1 by h_2 and b_{12} by b_{22}, respectively.

Hence, explicit expressions for all integrals w_{ik} are found in terms of tabulated functions.

6. Conclusion

This paper presents concrete variational formulas for self- and mutual impedances of two coupled antennas of unequal sizes, based on two-term Storer's trial current distribution. The three exitation conditions from which the impedances are calculated are choosen for the most simple evaluation, requiring only determination of three complex determinants of the second order. The impedances are expressed in terms of tabulated functions only. As in the case of a single antenna, the method is applicable for h_1, $h_2 \leqslant 3\,\lambda/4$.[1]

7. References

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