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TRIANGLE FUNCTIONAL EQUATION AND ITS GENERALIZATION*

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§1. Introduction. Let $M_1, \ldots, M_k$ be different points in Euclidean n-dimensional space E_n. We shall consider a linear homogeneous functional equation

$$\sum_{i=1}^{k} a_i f(M+tM_i)=0. \tag{1}$$

where a_i are real nonzero constants (weights), M is a variable point of E_n, t is a real independent variable, and f an unknown real-valued function. We assume that $f(M)$ is defined for all $M \in E_n$.

It is evident that (1) has no nontrivial continuous solution if $a_1+\cdots+a_k \neq 0$.

Let $\mathcal{A}$ be a regular affine transformation of E_n, and put $\mathcal{A}^{-1}M=N$, $\mathcal{A}^{-1}M_i=N_i$,

$$g(M)=f(\mathcal{A}M). \tag{2}$$

Then g satisfies the functional equation

$$\sum_{i=1}^{k} a_i g(N+tN_i)=0. \tag{3}$$

The equality (2) gives one-to-one correspondence between the solutions of (1) and (2). It enables us to obtain the solution of the functional equation (3) when the solution of equation (1) is known.

We are interested in some particular cases of the functional equation (1). These are

$$af(x, y)+bf(x+t, y)+cf(x, y+t)=f(x+rt, y+pt), \tag{4}$$

$$f(x-t, y)+f(x+t, y)+f(x, y+3t)=3f(x, y+t), \tag{5}$$

$$f(x-t, y)+f(x+t, y)+f(x, y+t\sqrt{3})=3f(x, y+t/\sqrt{3}), \tag{6}$$

$$2f(x, y)+f(x+t, y)+f(x, y+t)=4f(x+t/4, y+t/4). \tag{7}$$

The functional equation (6) is known as „triangle" equation. In this paper we prove (under weak regularity suppositions) that these equations have only

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polynomial solutions. We give explicit form of the solution of equations (5), (6), (7).

§ **2. Auxiliary results.** One-dimensional case of equation (1) has the form

$$\sum_{i=1}^{k} a_i f(x+\alpha_i t)=0. \tag{8}$$

Following T. Popoviciu [4] we define the characteristic „polynomial" of the equation (8)

$$F(x)=\sum_{i=1}^{k} a_i x^{\alpha_i}. \tag{9}$$

If $F(1)=F'(1)=\cdots=F^{(m-1)}(1)=0$, $F^{(m)}(1)\neq 0$ then we say that $F(x)$ has order m. These conditions are equivalent to

$$\sum_{i=1}^{k} a_i \alpha_i^{\nu}=0 \qquad (\nu=0,\,1,\,\ldots,\,m-1), \qquad \sum_{i=1}^{k} a_i \alpha_i^{m}\neq 0.$$

Consider the following transformations of the functional equation (8):(a) multiplication by a constant $a\neq 0$, (b) substitution $x\to x+\alpha t$, (c) substitution $t\to\beta t$ where $\beta\neq 0$. The corresponding transformations of $F(x)$ are: (a) $F(x)$ is multiplied by the same constant $a\neq 0$, (b) α_i are substituted by $\alpha+\alpha_i$, (c) x is substituted by x^{β} where $\beta\neq 0$. Characteristic „polynomials" which can be obtained one from another by application of a finite number of these transformations are said to be equivalent. It is easy to see that equivalent characteristic „polynomials" have the same order.

Let us define the operation $*$ by

$$f(x+\alpha t) * a x^{\beta} = a f(x+\alpha t+\beta t).$$

We extend this definition by assuming that the operation is linear in both factors. It may be verified that

$$\big(f(x+\alpha t) * F(x)\big) * G(x) = f(x+\alpha t) * \big(F(x)\, G(x)\big)$$

where $F(x)$ and $G(x)$ have the form (9). The equation (8) can be written in the form $f(x) * F(x)=0$.

The m-th difference is defined by

$$\Delta_h^m f(x)=\sum_{i=1}^{m} (-1)^i \binom{m}{i} f(x+ih).$$

The following theorem is due to T. Popoviciu [4] (p. 57):

Theorem 1. *If all α_i are rational, m is the order of* (9), *then equation* (8) *implies that*

$$\Delta_h^m f(x)=0, \tag{10}$$

where x, h are independent variables. Conversely, equation (10) *implies* (8).

Z. Ciesielski has proved [2] the following theorem:

Theorem 2. *Let $f(x)$ be defined over (a, b) and $\Delta_h^m f(x) \geqslant 0$, and let $f(x)$ be bounded on a set $E \subset (a, b)$ of positive measure. Then f is continuous on (a, b).*

By combining these two theorems we get:

Theorem 3. *Let $f(x)$ satisfy* (8) *with all a_i rational, and let $f(x)$ be bounded on a set of positive measure. Then $f(x)$ is continuous for all x.*

It is well known that the general continuous solution of the functional equation (10) is an arbitrary polynomial of degree less than m (cf. [4]).

Let μ and ν denote Lebesgue linear and plane measure respectively, and let $X(a) = \{(x, y) \mid x = a\}$, $Y(b) = \{(x, y) \mid y = b\}$.

Theorem 4. *Let $f(x, y)$ be defined for all x and y. Suppose that f has the following properties:*

(P_1) $f(x, y)$ is bounded on a set $S \subset E_2$ and $\nu(S) > 0$.

(P_2) If $f(x, y)$ is bounded on a set $A \subset X(a)$ with $\mu(A) > 0$ then $f(a, y($ is a polynomial in y of degree less than k.

(P_3) If $f(x, y)$ is bounded on a set $B \subset Y(b)$ with $\mu(B) > 0$ then $f(x, b)$ is a polynomial in x of degree less than l.

Then $f(x, y)$ is a polynomial in x and y whose degree in x resp. y is less than l resp. k.

Proof. Let us put

$$S_x = \{y \mid (x, y) \in S\}, \quad S_y = \{x \mid (x, y) \in S\},$$

$$X = \{x \mid \mu(S_x) > 0\}, \quad Y = \{y \mid \mu(S_y) > 0.$$

Property (P_1) and

$$\nu(S) = \int_{-\infty}^{+\infty} \mu(S_x)\, dx = \int_{-\infty}^{+\infty} \mu(S_y)\, dy$$

imply that $\mu(X) > 0$, $\mu(Y) > 0$. Let us choose different $a_i \in X$ $(i = 1, \ldots, l)$ and different $b_j \in Y$ $(j = 1, \ldots, k)$ and denote

$$P(x, y) = \sum_{i=1}^{l} \sum_{j=1}^{k} f(a_i, b_j) \left(\prod_{s}^{s \neq i} \frac{x - a_s}{a_i - a_s} \right) \left(\prod_{t}^{t \neq j} \frac{y - b_t}{b_j - b_t} \right).$$

Since $f(a_i, b_j) = P(a_i, b_j)$ for all i and j, properties (P_2) and (P_3) imply that

$$f(a_i, y) = P(a_i, y) \text{ for all } i \text{ and } y,$$

$$f(x, b_j) = P(x, b_j) \text{ for all } j \text{ and } x.$$

These equalities and properties (P_1) and (P_2) imply that

$$f(x, y) = P(x, y) \qquad \text{for } x \in X \text{ and all } y,$$

$$f(x, y) = P(x, y) \qquad \text{for } y \in Y \text{ and all } x.$$

Now, these equalities and the properties (P_1) and (P_2) imply that $f(x, y) = P(x, y)$ for all x and y.

Remark. *It is evident that a similar result holds if instead of $X(a)$ and $Y(b)$ we take any other two families of parallel straight lines.*

§ 3. **Statement of the results.** Let $D(a, b, c, p, r; x)$ be the determinant of the third order $|D_{ij}|$ whose elements are given by

$$D_{11} = ax^{qr} + bx^{p+pq+qr} - a(a+c) - b(b+c)x - abx^p - abx^q,$$

$$D_{12} = ax^{pr} + bx^{q+pr} - ax^r - bx^{r+pq}, \quad D_{13} = 0, \quad D_{21} = a + bx^p,$$

$$D_{22} = -x^{pr}, \quad D_{23} = c, \quad D_{31} = ax^{qr} + bx^{p+pq+qr} - a - bx,$$

$$D_{32} = a + bx^{q+pq} - ax^r - bx^{r+pq},$$

$$D_{33} = (a+b)x^r - ax^{qr} - bx^{pq+qr}.$$

Theorem 5. *We assume that*

1° *a, b, c are nonzero real constants such that $a + b + c = 1$.*

2° *p, r are rational constants different from 0 and 1, and $q = 1 - p$.*

3° *$f(x, y)$ has property (P_1) (see Theorem 4).*

4° *$f(x, y)$ satisfies the functional equation (4).*

Then $f(x, y)$ is continuous in the whole plane and it is a polynomial in x and y, whose degree in x resp. y is less than the order of $D(a, b, c, p, r; x)$ resp. $D(a, c, b, r, p; x)$. Moreover, these orders are at least 3.

By specialisation of Theorem 5 we obtain:

Theorem 6. *If $f(x, y)$ has property (P_1) and satisfies the functional equation (4) with $a = b = c = p = r = 1/3$, then $f(x, y)$ is a linear combination of the polynomials*

$$3x^3 - 3y^3 + (y-x)^3, \quad x(x+2y), \quad y(y+2x), \quad x, \ y, \ 1. \tag{11}$$

Conversely, any linear combination of these polynomials is a solution of this special equation (4).

Corollary 1. *If $f(x, y)$ has property (P_1) and satisfies the functional equation (5) then $f(x, y)$ is a linear combination of the polynomials*

$$x(x^2 - y^2), \quad 3x^2 - y^2, \quad xy, \ x, \ y, \ 1. \tag{12}$$

Corollary 2. *If $f(x, y)$ has property (P_1) and satisfies the functional equation (6) then $f(x, y)$ is a linear combination of the harmonic polynomials*

$$x^3 - 3xy^2, \quad x^2 - y^2, \quad xy, \ x, \ y, \ 1. \tag{13}$$

Theorem 7. *If $f(x, y)$ has property (P_1) and satisfies the functional equation (7) then $f(x, y)$ is a linear combination of the polynomials*

$$(x-y)(5x^2 + 14xy + 5y^2), \quad x(x+3y), \quad y(y+3x), \quad x, \ y, \ 1. \tag{14}$$

Conversely, any linear combination of these polynomials satisfies the functional equation (7).

For related results on the functional equation (1) see [1], [3]. Our proof is applicable to more general equations (4).

§ **4. Proof of Theorem 5.** For fixed y and fixed t we shall write

$$f(x, y) = \alpha(x), \quad f(x, y+p^2 t) = \beta(x),$$

$$f(x, y+p^2 t+p^2 qt) = \gamma(x), \quad f(x, y+pt) = \delta(x),$$

$$f(x, y+pt+p^2 qt) = \varepsilon(x),$$

$$f(x, y+pt+pqt) = \zeta(x), \quad f(x, y+t) = \eta(x).$$

Using the functional equation (4) we get

$$a\,\alpha(x) + b\,\alpha(x+t) + c\,\eta(x) = \delta(x+rt),$$

$$a\,\delta(x) + b\,\delta(x+qt) + c\,\eta(x) = \zeta(x+qrt),$$

$$a\,\alpha(x) + b\,\alpha(x+pt) + c\,\delta(x) = \beta(x+prt),$$

$$a\,\alpha(x) + b\,\alpha(x+pt+pqt) + c\,\zeta(x) = \gamma(x+prt+pqrt),$$

$$a\,\beta(x) + b\,\beta(x+pqt) + c\,\delta(x) = \gamma(x+pqrt),$$

$$a\,\beta(x) + b\,\beta(x+qt+pqt) + c\,\eta(x) = \varepsilon(x+qrt+pqrt),$$

$$a\,\delta(x) + b\,\delta(x+pqt) + c\,\zeta(x) = \varepsilon(x+pqrt).$$

By elimination of β, γ, δ, ε, ζ, η from this system of equations we obtain the functional equation satisfied by $\alpha(x)$. This equation has the form

$$\alpha(x) * F(x) = 0, \tag{15}$$

where $F(x)$ is the determinant

$$\begin{vmatrix} a+bx & 0 & -x^r & 0 & c \\ 0 & 0 & a+bx^q & -x^{qr} & c \\ a+bx^p & -x^{pr} & c & 0 & 0 \\ a+bx^{p+pq} & -ax^{pr}-bx^{pq+pr} & -cx^{pr} & c & 0 \\ 0 & a+bx^{q+pq} & -ax^{qr}-bx^{pq+qr} & -cx^{qr} & c \end{vmatrix}$$

Subtract the first row from the second and fıfth, multiply the second row by c, add the fourth row multiplied by x^{qr} to the second and fifth. We get (omitting a scalar factor)

$$\begin{vmatrix} ax^{qr}+bx^{p+pq+qr}-ac-bcx & -ax^r-bx^{r+pq} & c(a+bx^q) \\ a+bx^p & -x^{pr} & c \\ ax^{qr}+bx^{p+pq+qr}-a-bx & a+bx^{q+pq}-ax^r-bx^{r+pq} & (a+b)x^r-ax^{qr}-bx^{pq+qr} \end{vmatrix}.$$

Subtract the second row multiplied by $a+bx^q$ from the first. Then we get $D(a, b, c, p, r; x)$. Hence, $F(x)$ is equivalent to $D(x)$. We have derived (15) by assuming that t is a constant. Now we can consider t as a variable but y remains fixed.

Considering the elements in the second row of $D(x)$ we see that the columns of $D(x)$ are linearly independent. Hence, $F(x)$ is not identically zero. Since p, q, r are rational we may use Theorem 3. It follows that $f(x, y)$ has the property (P_3).

By symmetry (interchanging x and y, b and c, p and r) we conclude that $f(x, y)$ has the property (P_2) (see Theorem 4).

We have assumed (3°) that $f(x, y)$ has property (P_1). By Theorem 4, $f(x, y)$ is a polynomial whose degree in x resp. y is less than the order of $D(a, b, c, p, r; x)$ resp. $D(a, c, b, r, p; x)$.

It remains to prove that the order of $D(x)$ is at least 3. If we substitute $x=1$ in $D(x)$ then the first and the third row will be zero. Hence, $D(1)=D'(1)=0$. The derivative of the first row of $D(x)$ is proportional to $(1, -1,0)$. The derivative of the third row of $D(x)$ is proportional to $(0, -1,1)$. Since

$$\begin{vmatrix} 1 & -1 & 0 \\ a+b & -1 & c \\ 0 & -1 & 1 \end{vmatrix} = 0,$$

we conclude that $D''(1)=0$. This proves that the order of $D(x)$ is at least 3. We have

$$D'''(1)=6p^2q(ar+br-bq)(bcqr-acr^2-bcr^2+abc-abp).$$

So, $D'''(1)\neq 0$ in general.

Theorems 6 and 7 are particular cases of Theorem 5 in which we are able to determine the explicit form of the solution. This can be achieved by substituting each homogeneous polynomial separately. Corollaries 1 and 2 are obtained from Theorem 6 by applying appropriate affine transformations as described in § 1.

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