PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 188 (1967)

A TRIGONOMETRIC IDENTITY*

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(Received June 19, 1967)

We shall prove the following identity

$$\sum_{k=1}^{n} \cos^r \left(\varphi + \frac{2 k \pi}{n}\right) = \frac{n}{2^r} \sum \binom{r}{\nu} e^{i \lambda n \varphi} \tag{1}$$

where $r \geqslant 0$ and $n \geqslant 1$ are integers; the right-hand sum is taken over all integers ν, λ satisfying $0 \leqslant \nu \leqslant r$, $\lambda n + r = 2 \nu$.

Let us indicate some special cases of this identity.

(i) If $0 < r = 2 m < n$ we get

$$\sum_{k=1}^{n} \cos^{2 m} \left(\varphi + \frac{2 k \pi}{n}\right) = \frac{n}{4^m} \binom{2 m}{m}.$$

(ii) If $0 < r = 2 m - 1 < n$ we get

$$\sum_{k=1}^{n} \cos^{2 m-1} \left(\varphi + \frac{2 k \pi}{n}\right) = 0.$$

(iii) If $r = n = 2 m$ we get

$$\sum_{k=1}^{2 m} \cos^{2 m} \left(\varphi + \frac{k \pi}{m}\right) = \frac{2 m}{4^m} \left[\binom{2 m}{m} + 2 \cos 2 m \varphi\right].$$

(iv) If $r = n = 2 m - 1$ we get

$$\sum_{k=1}^{2m-1} \cos^{2 m-1} \left(\varphi + \frac{2 k \pi}{2 m-1}\right) = \frac{2 m-1}{4^{m-1}} \cos (2 m-1) \varphi.$$

The following identity of A. Friedman [1]

$$\sum_{k=1}^{n} \left(a \cos \frac{2 k \pi}{n} + b \sin \frac{2 k \pi}{n}\right)^r = \begin{cases} \dfrac{n}{4^m} \dbinom{2 m}{m} (a^2 + b^2)^m & (0 < r = 2 m < n) \\ 0 & (0 < r = 2 m - 1 < n) \end{cases}$$

* Presented by D. S. Mitrinović.

is a simple variant of identities (i) and (ii). We have only to set $a=\varrho\cos\varphi$, $b=-\varrho\sin\varphi$ with $\varrho=\sqrt{a^2+b^2}$.

Now we proceed to the proof of (1). Let m be a non-negative integer such that $mn\leqslant r<(m+1)\,n$ and

$$f(z)=\frac{nz^{n-1}}{z^n-e^{in\varphi}}\left(\frac{z^2+1}{2z}\right)^r\left(\frac{z^n}{e^{in\varphi}}\right)^m.$$

If we denote the left-hand side of (1) by $F_r^n(\varphi)$, then

$$F_r^n(\varphi)=\frac{1}{2\pi i}\oint\limits_{|z|=R} f(z)dz \qquad (R>1).$$

Using

$$\frac{nz^{n-1}}{z^n-e^{in\varphi}}=\frac{n}{z}\sum_{\mu=0}^{+\infty}\left(\frac{e^{in\varphi}}{z^n}\right)^\mu \qquad (|z|>1),$$

we obtain

$$F_r^n(\varphi)=\frac{n}{2\pi i}\cdot\frac{1}{2^r}\oint\limits_{|z|=R}\sum_{\nu=0}^{n}\binom{r}{\nu}z^{2\nu}\cdot\sum_{\lambda=-m}^{+\infty}\frac{e^{i\lambda n\varphi}}{z^{\lambda n+r+1}}\,dz$$

$$=\frac{n}{2^r}\sum_{\substack{\lambda n+r=2\nu\\ 0\leq\nu\leq r}}\binom{r}{\nu}e^{i\lambda n\varphi}.$$

The proof is complete.

REFERENCE

[1] A. Friedman, *Mean values and polyharmonic polynomials* Mich. Math. J. **4** (1957), 67—74.