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A RULE FOR THE CALCULATION OF MATRIX ELEMENTS OF THE ANGULAR MOMENTUM COMPONENTS

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Matrix elements that correspond to the operators $\hat{L}_x$ and $\hat{L}_y$ of the angular momentum components are usually calculated from

$$\hat{L}_x = \frac{1}{2}(\hat{A}^* + \hat{A}),$$
$$\hat{L}_y = \frac{i}{2}(\hat{A}^* - \hat{A}), \tag{1}$$

where $\hat{A}$ and $\hat{A}^*$ are defined as follows[1]:

$$A = \hbar\sqrt{(l-m)(l+m+1)}\cdot\delta_{m,m+1}$$
$$A^* = \hbar\sqrt{(l+m)(l-m+1)}\cdot\delta_{m,m-1}. \tag{2}$$

The only elements different from zero are the off-diagonal elements (the upper ones for A and the lower ones for A^*).

Every matrix element has therefore to be calculated separately.

Let us consider the matrix elements of A.

Denote

$$\frac{A_{12}^2}{\hbar^2} = a_{12}, \quad \ldots, \quad \frac{A_{k,k+1}^2}{\hbar^2} = a_{k,k+1}. \tag{3}$$

Then

$$\begin{aligned} a_{12} &= 2l\cdot 1 \\ a_{23} &= (2l-1)\cdot 2 \\ a_{34} &= (2l-2)\cdot 3 \\ &\vdots \\ a_{k,k+1} &= (2l+1-k)\cdot k. \end{aligned} \tag{4}$$

These are the matrix elements of a $(2l+1)\times(2l+1)$ matrix. The difference between two consecutive elements, the first for l and the second for $l+\frac{1}{2}$, depends only of the given l-value.

[1] See e.g. A. R. Edmonds, Angular Momentum in Quantum Mecanics, 1963.

One obtains

(5) $$\underset{(l+1/2)}{a_{k,k+1}} - \underset{(l)}{a_{k,k+1}} = k(2l+1+1-k) - k(2l+1-k) = k.$$

Thus:

For a_{12}:	For a_{23}:	For a_{34}:
$a_{12(l=1/2)} = 1$		
(6) $a_{12(l=1)} = 2$	$a_{23(l=1)} = 2$	
$a_{12(l=3/2)} = 3$	$a_{23(l=3/2)} = 4$	$a_{34(l=3/2)} = 3$
$\vdots$	$a_{23(l=2)} = 6$	$a_{34(l=2)} = 6$
	$\vdots$	$a_{34(l=5/2)} = 9$
		$\vdots$

Accordingly, a very simple rule can be formulated, that gives directly *the squares of the matrix elements arranged in the form of a triangle* (or table).

The first column contains integers 1, 2, 3, ... ; the second is the first column multiplied by 2 (the even numbers): 2, 4, 6, ... ; the third — multiplied by 3: 3, 6, 9, ... , and so forth, as follows

(7)

$(l=1/2)$	1				
$(l=1)$	2	2			
$(l=3/2)$	3	4	3		
$(l=2)$	4	6	6	4	
$(l=5/2)$	5	8	9	8	5
$\vdots$					

The first column represents the numbers $2l$, the second: $(2l-1)\cdot 2, \ldots$, the k-th column the numbers $(2l+1-k)\cdot k$.

From here the squares of all off-diagonal matrix elements of the matrices A and A^* (and thus of L_x and L_y) one reads directly in the corresponding row for a given l, in the form:

(8) $$2l\cdot 1,\ (2l-1)\cdot 2,\ (2l-2)\cdot 3,\ \ldots,\ 3\cdot(2l-2),\ 2\cdot(2l-1),\ 1\cdot 2l.$$

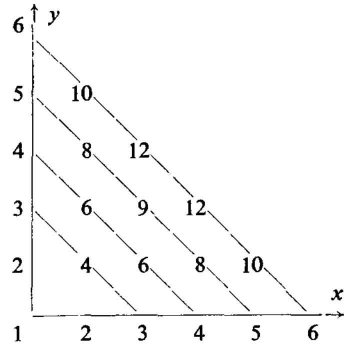
This form can be useful especially for larger l-values. E.g.:

$l=4$ $(9\times 9$ matrix)

$a_{12} = 8\cdot 1 = 8$	$a_{56} = 20$
$a_{23} = 7\cdot 2 = 14$	$a_{67} = 18$
$a_{34} = 6\cdot 3 = 18$	$a_{78} = 14$
$a_{45} = 5\cdot 4 = 20$	$a_{89} = 8.$

Geometrically a more illustrative form can be given to (7) writing it:

(9)



where the squares of the matrix elements (beeng simply the product of the "coordinates" x and y) are already arranged along the diagonals (the hypothenuse of the triangle).

Example.

For $l = 5/2$, from (7), (8) or (9) one can directly write down:

$$L_x = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & \sqrt{8} & 0 & \sqrt{9} & 0 & 0 \\ 0 & 0 & \sqrt{9} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & \sqrt{8} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{bmatrix}$$

$$L_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i\sqrt{5} & 0 & 0 & 0 & 0 \\ i\sqrt{5} & 0 & -i\sqrt{8} & 0 & 0 & 0 \\ 0 & i\sqrt{8} & 0 & -i\sqrt{9} & 0 & 0 \\ 0 & 0 & i\sqrt{9} & 0 & -i\sqrt{8} & 0 \\ 0 & 0 & 0 & i\sqrt{8} & 0 & -i\sqrt{5} \\ 0 & 0 & 0 & 0 & i\sqrt{5} & 0 \end{bmatrix}$$

avoiding completely the intermediate matrices A and A^*.

Because of its simplicity this triangle rule might be usefully applied in atomic and nuclear spectroscopy, as also generally in Quantum mecanics.