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AN OLD INEQUALITY REDISCOVERED BY WILF

Dragoslav S. Mitrinović

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This short note contains the history of inequality (1), *which seems to be due to serbian mathematician M. Petrović. This history shows once more that the rediscovery of a forgotten fact can initiate a discovery of new results.*

Theorem 1. *Let* a *be a real number and let* $0<\theta<\pi/2$. *If* $z_1, \ldots, z_n$ *are complex numbers such that*

$$a-\theta\leqslant\arg z_\nu\leqslant a+\theta \qquad (\nu=1, \ldots, n),$$

then

$$\left|\sum_{\nu=1}^{n} z_\nu\right|\geqslant(\cos\theta)\sum_{\nu=1}^{n}|z_\nu|. \tag{1}$$

Proof. We have

$$\left|\sum_{\nu=1}^{n} z_\nu\right|=\left|e^{-ia}\sum_{\nu=1}^{n} z_\nu\right|\geqslant \operatorname{Re}\left(e^{-ia}\sum_{\nu=1}^{n} z_\nu\right)$$

$$=\sum_{\nu=1}^{n}|z_\nu|\cos(-a+\arg z_\nu)\geqslant(\cos\theta)\sum_{\nu=1}^{n}|z_\nu|.$$

This proof is due to D. Ž. Đoković.

Inequality (1) is a complementary triangle inequality. It is difficult to say where it appears for the first time in the literature. We have found that the special case $a=\theta=\pi/4$ was proved by M. Petrovitch [1] in 1917. The general case of this inequality appears in a later paper [2] of M. Petrovitch. He also applied (1) to derive some inequalities for integrals. Inequality (1) can be found also in Karamata's book [3] (p. 300—301). In [3] one can find the following proposition:

Theorem 2. *If* f *is a complex-valued integrable function defined in the interval* $a\leqslant x\leqslant b$ *and*

$$-\theta\leqslant\arg f(x)\leqslant+\theta \qquad \left(0<\theta<\frac{\pi}{2}\right),$$

then

$$\left|\int_a^b f(x)\,\mathrm{d}x\right|\geqslant(\cos\theta)\cdot\int_a^b|f(x)|\,\mathrm{d}x.$$

Karamata has also published inequality (1) in his another book [4], (p. 155).

Old inequality (1) has been rediscovered by Wilf [5] in 1963. The generalization of (1) to Hilbert and Banach spaces has been given recently by Diaz and Metcalf [6]. We quote their theorems holding in any Hilbert space H:

Theorem 3. *Let a be a unit vector in H. Suppose the vectors $x_1, \ldots, x_n$, satisfy*

$$0 \leqslant r \leqslant \frac{\operatorname{Re}(x_i, a)}{|x_i|} \qquad (i=1, \ldots, n),$$

whenever $x_i \neq 0$. Then

$$r(|x_1| + \cdots + |x_n|) \leqslant |x_1 + \cdots + x_n|,$$

where equality holds if and only if

$$x_1 + \cdots + x_n = r(|x_1 + \cdots + |x_n|)\, a.$$

Theorem 4. *Let $a_1, \ldots, a_m$ be orthonormal vectors in H. Suppose the vectors $x_1, \ldots, x_n$ satisfy*

$$0 \leqslant r_k \leqslant \frac{\operatorname{Re}(x_i, a_k)}{|x_i|} \qquad (i=1, \ldots, n;\ k=1, \ldots, m),$$

whenever $x_i \neq 0$. Then

$$(r_1^2 + \cdots + r_m^2)^{1/2} (|x_1| + \cdots + |x_n|) \leqslant |x_1 + \cdots + x_n|,$$

where equality holds if and only if

$$x_1 + \cdots + x_n = (|x_1| + \cdots + |x_n|)(r_1 a_1 + \cdots + r_m a_m).$$

The following variant of (1) for $\theta = \pi/2$ appears in Marden [7] (p. 1): *If each complex number z_ν $(\nu = 1, \ldots, n)$ has the properties that $z_\nu \neq 0$ and*

$$\alpha \leqslant \arg z_\nu < \alpha + \pi,$$

then their sum $z_1 + \cdots + z_n$ cannot vanish.

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See also:

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