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SOME INEQUALITIES INVOLVING ELEMENTARY SYMMETRIC FUNCTIONS

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1. ***Introduction.*** Let a_k $(k=1, \ldots, n)$ be real numbers, and consider

$$f(x) = \prod_{k=1}^{n} (x+a_k) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n. \tag{1.1}$$

Then c_k $(k=1, \ldots, n)$ is the k-th elementary symmetric function of the a_k $(k=1, \ldots, n)$. We define $c_0=1$ and $c_{-k}=0$ $(k=1, 2, \ldots)$.

The following result is valid (see [1], p. 52 or [2], p. 117):

Theorem 1.1. *If all zeros of a real polynomial $f(x)$ are real, then*

$$c_{k-1} c_{k+1} - c_k^2 \leqslant 0 \qquad (k=1, \ldots, n-1). \tag{1.2}$$

Applying this theorem to the polynomial

$$(x-1)^{\nu} f(x) \qquad (\nu \text{ a positive integer}),$$

we get the following inequality

$$\Big(\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} c_{k-1-i}\Big) \Big(\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} c_{k+1-i}\Big) - \Big(\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} c_{k-i}\Big)^2 \leqslant 0 \qquad (k=1, \ldots, n-1). \tag{1.3}$$

For instance, if $\nu=1$ and $\nu=2$, this inequality becomes

$$(c_{k-1}-c_{k-2})(c_{k+1}-c_k) \leqslant (c_k-c_{k-1})^2 \tag{1.4}$$

and

$$(c_{k-1}-2c_{k-2}+c_{k-3})(c_{k+1}-2c_k+c_{k-1}) \leqslant (c_k-2c_{k-1}+c_{k-2})^2, \tag{1.5}$$

respectively.

2. ***Preliminary results.*** Let $c_{k+1}>c_k>0$. Then inequality (1.2) is equivalent to

$$\frac{c_{k-1}}{c_k}\leqslant\frac{c_k}{c_{k+1}},$$

whence

$$\frac{c_{k-1}}{c_k}\leqslant\frac{c_k}{c_{k+1}}<1,$$

since $c_k<c_{k+1}$. Thus,

Theorem 2.1. *If* $k=1,\ldots,n-1$, *then* $c_{k+1}>c_k>0 \ \Rightarrow\ c_k>c_{k-1}$.

Similarly, we obtain the result:

Theorem 2.2. *If* $k=1,\ldots,n-1$, *then* $c_{k-1}>c_k>0 \ \Rightarrow\ c_k>c_{k+1}$.

The following theorems are also valid:

Theorem 2.3. *If* $k=1,\ldots,n-1$, *then*

$$c_k>c_{k-1}>0 \ \textit{and}\ c_{k+1}-2\,c_k+c_{k-1}>0$$

imply that

$$c_k-2\,c_{k-1}+c_{k-2}>0.$$

Theorem 2.4. *If* $k=1,\ldots,n-1$, *then*

$$0<c_k<c_{k-1} \ \textit{and}\ c_k-2\,c_{k-1}+c_{k-2}>0$$

imply that

$$c_{k+1}-2\,c_k+c_{k-1}>0.$$

Theorems 2.1—2.4 are simple generalizations of theorems 1 and 2 proved by J. N. Darroch and J. Pitman [3].

We shall prove only theorem 2.4. Consider the inequality (1.4). Since $c_k-c_{k-1}<0$ by our assumption and $c_{k+1}-c_k<0$ by theorem 2.2, inequality (1.4) is equivalent to

$$\frac{c_{k-1}-c_{k-2}}{c_k-c_{k-1}}\leqslant\frac{c_k-c_{k-1}}{c_{k+1}-c_k}. \tag{2.1}$$

We have assumed also that

$$c_k-2\,c_{k-1}+c_{k-2}>0,\ \text{i.e.,}\ c_k-c_{k-1}>c_{k-1}-c_{k-2},$$

so that

$$1<\frac{c_{k-1}-c_{k-2}}{c_k-c_{k-1}}.$$

From (2.1) and the last inequality we get

$$1<\frac{c_k-c_{k-1}}{c_{k+1}-c_k}.$$

Since $c_{k+1}-c_k<0$, the latter is equivalent to

$$c_{k+1}-c_k>c_k-c_{k-1}, \text{ i.e., } c_{k+1}-2c_k+c_{k-1}>0,$$

which we had to prove.

Next we prove the following two preliminary theorems:

Theorem 2.5. *If*

$$c_{k+1}-3c_k+3c_{k-1}-c_{k-2}>0, \tag{2.2}$$

$$c_k-2c_{k-1}+c_{k-2}>0, \tag{2.3}$$

$$c_{k-1}-c_{k-2}>0, \tag{2.4}$$

and

$$c_{k-2}>0, \tag{2.5}$$

then

$$c_k-3c_{k-1}+3c_{k-2}-c_{k-3}>0. \tag{2.6}$$

Proof. Starting with (1.5) we have

$$\frac{c_{k+1}-2c_k+c_{k-1}}{c_k-2c_{k-1}+c_{k-2}}\leqslant\frac{c_k-2c_{k-1}+c_{k-2}}{c_{k-1}-2c_{k-2}+c_{k-3}}, \tag{2.7}$$

since, by (2.3), (2.4) and (2.5),

$$c_k-2c_{k-1}+c_{k-2}>0 \text{ and } c_{k-1}-2c_{k-2}+c_{k-3}>0. \tag{2.8}$$

From (2.2) it follows that

$$c_{k+1}-2c_k+c_{k-1}>c_k-2c_{k-1}+c_{k-2},$$

i. e.,

$$\frac{c_{k+1}-2c_k+c_{k-1}}{c_k-2c_{k-1}+c_{k-2}}>1.$$

From this result and (2.7), we obtain

$$1<\frac{c_k-2c_{k-1}+c_{k-2}}{c_{k-1}-2c_{k-2}+c_{k-3}}.$$

Since $c_{k-1}-2c_{k-2}+c_{k-3}>0$, the latter is equivalent to

$$c_{k-1}-2c_{k-2}+c_{k-3}<c_k-2c_{k-1}+c_{k-2},$$

i. e.,

$$0<c_k-3c_{k-1}+3c_{k-2}-c_{k-3},$$

whence the truth of (2.6) follows.

Theorem 2.6. *If*

(2.9) $$c_k - 3\,c_{k-1} + 3\,c_{k-2} - c_{k-3} < 0,$$

(2.10) $$c_k - 2\,c_{k-1} + c_{k-2} > 0,$$

(2.11) $$c_k - c_{k-1} < 0,$$

and

(2.12) $$c_k > 0,$$

then

(2.13) $$c_{k+1} - 3\,c_k + 3\,c_{k-1} - c_{k-2} < 0.$$

Proof. We start with inequality (1.5). By theorem 2.4 and hypotheses (2.10), (2.11) and (2.12) we have

$$c_{k+1} - 2\,c_k + c_{k-1} > 0 \text{ and } c_k - 2\,c_{k-1} + c_{k-2} > 0.$$

Then inequality (1.5) is equivalent to

(2.14) $$\frac{c_{k-1} - 2\,c_{k-2} + c_{k-3}}{c_k - 2\,c_{k-1} + c_{k-2}} \leqslant \frac{c_k - 2\,c_{k-1} + c_{k-2}}{c_{k+1} - 2\,c_k + c_{k-1}}.$$

Inequality (2.9) is equivalent to

$$c_k - 2\,c_{k-1} + c_{k-2} < c_{k-1} - 2\,c_{k-2} + c_{k-3};$$

i. e.,

(2.15) $$1 < \frac{c_{k-1} - 2\,c_{k-2} + c_{k-3}}{c_k - 2\,c_{k-1} + c_{k-2}}.$$

Inequalities (2.14) and (2.15) yield

$$1 < \frac{c_k - 2\,c_{k-1} + c_{k-2}}{c_{k+1} - 2\,c_k + c_{k-1}},$$

from which (2.13) follows immediately.

3. ***Main results.*** The results mentioned in § 2 suggest that some more general theorems might hold.

We shall now use the notation

$$\Delta^m c_r = \sum_{i=0}^{m} (-1)^i \binom{m}{i} c_{r+i}.$$

Theorem 3.1. *Let k and ν be fixed natural numbers and, for $p = 0, 1, \ldots, \nu$, let*

(3.1) $$(-1)^p \Delta^p c_{k-\nu+1} > 0;$$

then

(3.2) $$(-1)^\nu \Delta^\nu c_{k-\nu} > 0.$$

Proof. For $\nu=1, 2, 3$ and each $k=1, \ldots, n-1$ theorem 3.1 reduces to theorems 2.1, 2.3, 2.5. Let us assume that theorem 3.1 holds for $\nu-1$ and all $k=1, \ldots, n-1$; i. e. that the inequalities

$$(3.3) \qquad (-1)^p \Delta^p c_{k-\nu+1} > 0 \qquad (p=0, 1, \ldots, \nu-1)$$

imply that

$$(3.4) \qquad (-1)^{\nu-1} \Delta^{\nu-1} c_{k-\nu} > 0.$$

In this case inequality (1.3) becomes

$$(3.5) \qquad \Delta^{\nu-1} c_{k-\nu} \cdot \Delta^{\nu-1} c_{k-\nu+2} \leqslant (\Delta^{\nu-1} c_{k-\nu+1})^2.$$

On the basis of (3.2) for $p=\nu-1$ and (3.4), inequality (3.5) is equivalent to

$$(3.6) \qquad \frac{\Delta^{\nu-1} c_{k-\nu+2}}{\Delta^{\nu-1} c_{k-\nu+1}} \leqslant \frac{\Delta^{\nu-1} c_{k-\nu+1}}{\Delta^{\nu-1} c_{k-\nu}}.$$

Inequality (3.1) for $p=\nu$ is equivalent to

$$(-1)^{\nu-1} \Delta^{\nu-1} c_{k-\nu+2} > (-1)^{\nu-1} \Delta^{\nu-1} c_{k-\nu+1},$$

wherefrom it follows that

$$(3.7) \qquad 1 < \frac{\Delta^{\nu-1} c_{k-\nu+2}}{\Delta^{\nu-1} c_{k-\nu+1}}.$$

From (3.6) and (3.7) we obtain

$$1 < \frac{\Delta^{\nu-1} c_{k-\nu+1}}{\Delta^{\nu-1} c_{k-\nu}}.$$

This inequality is equivalent to

$$(-1)^{\nu-1} \Delta^{\nu-1} c_{k-\nu} < (-1)^{\nu-1} \Delta^{\nu-1} c_{k-\nu+1},$$

whence we obtain the inequality (3.2). This proves the theorem 3.1.

We can also prove the following theorems:

Theorem 3.2. *If s and k are natural numbers and*

$$\Delta^p c_{k-p} > 0 \qquad (p=0, 1, \ldots, 2s),$$

then

$$\Delta^{2s} c_{k-2s+1} > 0.$$

Theorem 3.3. *If s and k are natural numbers and*

$$\Delta^p c_{k-p} > 0 \qquad (p=0, 1, \ldots, 2s+1),$$

then

$$\Delta^{2s+1} c_{k-2s} > 0.$$

Proof. The proof of these theorems is by induction using the following schema:

Let $P(\nu)$ denote some property. If

$$P(2s-1) \Rightarrow P(2s),$$

and

$$P(2s) \Rightarrow P(2s+1),$$

then also

$$P(\nu) \Rightarrow P(\nu+1).$$

Let us assume that the inequalities

$$\Delta^p c_{k-p} > 0 \qquad (p=0, 1, \ldots, 2s-1)$$

imply $\Delta^{2s-1} c_{k-2s+2} > 0$. Now, if $\Delta^{2s} c_{k-2s} > 0$, we shall prove that $\Delta^{2s} c_{k-2s+1} > 0$.

In order to prove this fact, we start with

$$\Delta^{2s-1} c_{k-2s} \cdot \Delta^{2s-1} c_{k-2s+2} \leqslant (\Delta^{2s-1} c_{k-2s+1})^2.$$

This inequality is equivalent to

$$\frac{\Delta^{2s-1} c_{k-2s}}{\Delta^{2s-1} c_{k-2s+1}} \leqslant \frac{\Delta^{2s-1} c_{k-2s+1}}{\Delta^{2s-1} c_{k-2s+2}}. \tag{3.8}$$

The inequality $\Delta^{2s} c_{k-2s} > 0$ is equivalent to the following

$$\Delta^{2s-1} c_{k-2s+1} < \Delta^{2s-1} c_{k-2s};$$

i. e.,

$$1 < \frac{\Delta^{2s-1} c_{k-2s}}{\Delta^{2s-1} c_{k-2s+1}}. \tag{3.9}$$

From (3.8) and (3.9) we deduce

$$1 < \frac{\Delta^{2s-1} c_{k-2s+1}}{\Delta^{2s-1} c_{k-2s+2}},$$

whence

$$\Delta^{2s-1} c_{k-2s+2} < \Delta^{2s-1} c_{k-2s+1}, \text{ i. e., } \Delta^{2s} c_{k-2s+1} > 0.$$

Now suppose that the inequalities

$$\Delta^p c_{k-p} > 0 \qquad (p=0, 1, \ldots, 2s)$$

imply $\Delta^{2s} c_{k-2s+1} > 0$. If $\Delta^{2s+1} c_{k-2s-1} > 0$ it can be proved by the similar arguments that $\Delta^{2s+1} c_{k-2s} > 0$.

Hence, by the above arguments and theorems 2.4 and 2.6, theorems 3.2 and 3.3 are proved inductively.

4. ***Generalization.*** We can apply our procedure to

$$(x-a_1)(x-a_2)\cdots(x-a_\nu)(c_0x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n)$$
$$=A_0x^{n+\nu}+A_1x^{n+\nu-1}+\cdots+A_{n+\nu-1}x+A_{n+\nu},$$

where $a_1, \ldots, a_\nu$ are real numbers to obtain more general results than those we have proved.

It is also possible to modify the systems of conditional inequalities in § 2 and § 3.

5. ***Remark.*** Let us consider the polynomial

$$(x-a)(c_0x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n),$$

where a is a real number. Then inequality (1.2) becomes

$$(c_{k-1}-a\,c_{k-2})(c_{k+1}-a\,c_k)\leqslant(c_k-a\,c_{k-1})^2;$$

i. e., for every real a the following inequality is valid

$$(c_{k-2}c_k-c_{k-1}^2)a^2+(c_{k-1}c_k-c_{k-2}c_{k-1})a+(c_{k-1}c_{k+1}-c_k^2)\leqslant 0, \tag{5.1}$$

where, according to (1.2),

$$c_{k-2}c_k-c_{k-1}^2\leqslant 0 \qquad (k=1,\ldots,n-1). \tag{5.2}$$

From (5.1) and (5.2) we get

$$(c_{k-1}c_k-c_{k-2}c_{k+1})^2-4(c_{k-1}c_{k+1}-c_k^2)(c_{k-2}c_k-c_{k-1}^2)\leqslant 0.$$

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