PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 182 (1967)

CERTAIN INEQUALITIES FOR ELEMENTARY SYMMETRIC FUNCTIONS

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(Received June 1, 1967)

1. Let a_k $(k=1, \ldots, n)$ be real numbers and let

$$f(x)=(x+a_1) \cdots (x+a_n)=x^n+c_1 x^{n-1}+\cdots+c_{n-1} x+c_n.$$

Then c_k $(k=1, \ldots, n)$ is the k-th elementary symmetric function of $a_1, \ldots, a_n$. We define $c_0=1$ and $c_{-k}=0$ for $k=1, 2, \ldots$

Let $\overline{c}_k$ denote the k-th elementary symmetric function of $a_2, \ldots, a_n$.

The following inequality is valid for the coefficients of the polynomial $f(x)$ (see: [1], p. 117):

$$c_{k-1} c_{k+1}-c_k^2 \leqslant 0. \tag{1.1}$$

Let ν be a positive integer and consider the polynomial

$$\begin{aligned} F(x) &= (x-1)^\nu (c_0 x^n+c_1 x^{n-1}+\cdots+c_{n-1} x+c_n) \\ &= b_0 x^{n+\nu}+b_1 x^{n+\nu-1}+\cdots+b_{n+\nu-1} x+b_{n+\nu}. \end{aligned}$$

The inequality (1.1) corresponding to the polynomial $F(x)$ is

$$b_{k-1} b_{k+1}-b_k^2 \leqslant 0 \qquad (k=1, \ldots, n+\nu-1);$$

i.e.,

$$\left(\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i} c_{k-1-i}\right)\left(\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i} c_{k+1-i}\right)-\left(\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i} c_{k-i}\right)^2 \leqslant 0, \tag{1.2}$$

for $k=1, \ldots, n-1$.

In what follows $a_1, \ldots, a_n$ will denote positive numbers.

2. We begin with the identity

$$c_k=a_1 \overline{c}_{k-1}+\overline{c}_k$$

and consider

$$g(a_1)=c_{k+1}-c_k=a_1(\overline{c}_k-\overline{c}_{k-1})+(\overline{c}_{k+1}-\overline{c}_k).$$

This yields

Theorem 2.1. *If* $k=1, \ldots, n-2$, *then*

$$\bar{c}_k-\bar{c}_{k-1}>0 \Rightarrow c_{k+1}-c_k \geqslant \bar{c}_{k+1}-\bar{c}_k,$$

$$\bar{c}_k-\bar{c}_{k-1}<0 \Rightarrow c_{k+1}-c_k \leqslant \bar{c}_{k+1}-\bar{c}_k.$$

The following results are also valid:

Theorem 2.2. *For* $k=1, \ldots, n-1$ *the following inequality holds*

$$c_{k-1}c_{k+1}-c_k^2 \leqslant \bar{c}_{k-1}\bar{c}_{k+1}-\bar{c}_k^2.$$

Theorem 2.3. *If* $\bar{c}_{k-1}-\bar{c}_{k-2}>0$, *then*

$$(c_{k-1}-c_{k-2})(c_{k+1}-c_k)-(c_k-c_{k-1})^2 \leqslant (\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_{k+1}-\bar{c}_k)-(\bar{c}_k-\bar{c}_{k-1})^2.$$

Proof of theorem 2.3. By Darroch — Pitman's result [2]

$$c_{k+1}-c_k \geqslant 0 \Rightarrow c_k-c_{k-1}>0 \qquad (k=1, \ldots, n-1)$$

we have, by analogy,

$$\bar{c}_k-\bar{c}_{k-1} \geqslant 0 \Rightarrow \bar{c}_{k-1}-\bar{c}_{k-2}>0 \Rightarrow \bar{c}_{k-2}-\bar{c}_{k-3}>0. \tag{2.1}$$

Now consider the functions

$$\begin{aligned} h(a_1) &= (c_{k-1}-c_{k-2})(c_{k+1}-c_k)-(c_k-c_{k-1})^2 \\ &= [a_1(\bar{c}_{k-2}-\bar{c}_{k-3})+(\bar{c}_{k-1}-\bar{c}_{k-2})][a_1(\bar{c}_k-\bar{c}_{k-1})+(\bar{c}_{k+1}-\bar{c}_k)] \\ &\quad -[a_1(\bar{c}_{k-1}-\bar{c}_{k-2})+(\bar{c}_k-\bar{c}_{k-1})]^2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} h'(a_1) &= 2a_1[(\bar{c}_{k-2}-\bar{c}_{k-3})(\bar{c}_k-\bar{c}_{k-1})-(\bar{c}_{k-1}-\bar{c}_{k-2})^2] \\ &\quad +(\bar{c}_{k-2}-\bar{c}_{k-3})(\bar{c}_{k+1}-\bar{c}_k)-(\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_k-\bar{c}_{k-1}), \end{aligned} \tag{2.3}$$

$$h''(a_1)=2[(\bar{c}_{k-2}-\bar{c}_{k-3})(\bar{c}_k-\bar{c}_{k-1})-(\bar{c}_{k-1}-\bar{c}_{k-2})^2]. \tag{2.4}$$

For $\nu=1$ inequality (1.2) has the form

$$(c_{k-1}-c_{k-2})(c_{k+1}-c_k)-(c_k-c_{k-1})^2 \leqslant 0. \tag{2.5}$$

From this inequality it follows that $h''(a_1) \leqslant 0$ and consequently $h'(a_1) \leqslant h'(0)$.

From (2.3)

$$h'(0)=(\bar{c}_{k-2}-\bar{c}_{k-3})(\bar{c}_{k+1}-\bar{c}_k)-(\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_k-\bar{c}_{k-1}) \tag{2.6}$$

is obtained.

Starting with (2.5), we have

$$(\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_{k+1}-\bar{c}_k) \leqslant (\bar{c}_k-\bar{c}_{k-1})^2. \tag{2.7}$$

If $\bar{c}_k-\bar{c}_{k-1}\geqslant 0$, then, by virtue of (2.1), the inequality (2.7) is equivalent to

$$(\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_{k+1}-\bar{c}_k)(\bar{c}_{k-2}-\bar{c}_{k-3})\leqslant(\bar{c}_k-\bar{c}_{k-1})^2(\bar{c}_{k-2}-\bar{c}_{k-3}).$$

By (2.1) and (2.5) we obtain

$$(\bar{c}_{k-1}-\bar{c}_{k-2})(\bar{c}_{k+1}-\bar{c}_k)(\bar{c}_{k-2}-\bar{c}_{k-3})\leqslant(\bar{c}_k-\bar{c}_{k-1})(\bar{c}_{k-1}-\bar{c}_{k-2})^2.$$

Now, by virtue of (2.1) we have

$$(\bar{c}_{k+1}-\bar{c}_k)(\bar{c}_{k-2}-\bar{c}_{k-3})\leqslant(\bar{c}_k-\bar{c}_{k-1})(\bar{c}_{k-1}-\bar{c}_{k-2}). \tag{2.8}$$

From (2.8) we conclude that $h'(0)\leqslant 0$, and consequently $h'(a_1)\leqslant 0$. Hence $h(a_1)\leqslant h(0)$, which proves theorem 2.3.

3. Let us now prove our main result. We shall use the notation

$$\Delta^m c_r=\sum_{i=0}^{m}(-1)^i\binom{m}{i}c_{r+i}.$$

Theorem 3.1. *If*

$$(-1)^\nu\Delta^\nu\bar{c}_{k-\nu-2}>0,\quad(-1)^\nu\Delta^\nu\bar{c}_{k-\nu-1}>0\quad\textit{and}\quad(-1)^\nu\Delta^\nu\bar{c}_{k-\nu}>0, \tag{3.1}$$

then

$$\Delta^\nu c_{k-\nu-1}\,\Delta^\nu c_{k-\nu+1}-(\Delta^\nu c_{k-\nu})^2\leqslant\Delta^\nu\bar{c}_{k-\nu-1}\,\Delta^\nu\bar{c}_{k-\nu+1}-(\Delta^\nu\bar{c}_{k-\nu})^2.$$

Proof. Consider the function of a_1:

$$\begin{aligned}H(a_1)&=\Delta^\nu c_{k-\nu-1}\,\Delta^\nu c_{k-\nu+1}-(\Delta^\nu c_{k-\nu})^2\\&=(a_1\Delta^\nu\bar{c}_{k-\nu-2}+\Delta^\nu\bar{c}_{k-\nu-1})(a_1\Delta^\nu\bar{c}_{k-\nu}+\Delta^\nu\bar{c}_{k-\nu+1})\\&\quad-(a_1\Delta^\nu\bar{c}_{k-\nu-1}+\Delta^\nu\bar{c}_{k-\nu})^2.\end{aligned}$$

The derivatives of $H(a_1)$ are:

$$\begin{aligned}H'(a_1)&=\Delta^\nu\bar{c}_{k-\nu-2}(a_1\Delta^\nu\bar{c}_{k-\nu}+\Delta^\nu\bar{c}_{k-\nu+1})\\&\quad+\Delta^\nu\bar{c}_{k-\nu}(a_1\Delta^\nu\bar{c}_{k-\nu-2}+\Delta^\nu\bar{c}_{k-\nu-1})\\&\quad-2\Delta^\nu\bar{c}_{k-\nu-1}(a_1\Delta^\nu\bar{c}_{k-\nu-1}+\Delta^\nu\bar{c}_{k-\nu}),\end{aligned}$$

and

$$H''(a_1)=2(\Delta^\nu\bar{c}_{k-\nu-2}\Delta^\nu\bar{c}_{k-\nu}-(\Delta^\nu\bar{c}_{k-\nu-1})^2).$$

From (1.2) we have $H''(a_1)\leqslant 0$, and consequently

$$H'(a_1)\leqslant H'(0), \tag{3.2}$$

where

$$\begin{aligned}H'(0)&=\Delta^\nu\bar{c}_{k-\nu-2}\Delta^\nu\bar{c}_{k-\nu+1}+\Delta^\nu\bar{c}_{k-\nu}\Delta^\nu\bar{c}_{k-\nu-1}-2\Delta^\nu\bar{c}_{k-\nu-1}\Delta^\nu\bar{c}_{k-\nu}\\&=\Delta^\nu\bar{c}_{k-\nu-2}\Delta^\nu\bar{c}_{k-\nu+1}-\Delta^\nu\bar{c}_{k-\nu-1}\Delta^\nu\bar{c}_{k-\nu}.\end{aligned}$$

If conditions (3.1) are satisfied, by inequality (1.2) we have the following

$$\Delta^\nu \overline{c}_{k-\nu-1} \Delta^\nu \overline{c}_{k-\nu+1} \cdot (-1)^\nu \Delta^\nu \overline{c}_{k-\nu-2} \leqslant (\Delta^\nu \overline{c}_{k-\nu})^2 \cdot (-1)^\nu \Delta^\nu \overline{c}_{k-\nu-2}$$
$$\leqslant (-1)^\nu \Delta^\nu \overline{c}_{k-\nu} (\Delta^\nu \overline{c}_{k-\nu-1})^2,$$

whence

$$\Delta^\nu \overline{c}_{k-\nu+1} \Delta^\nu \overline{c}_{k-\nu-2} \leqslant \Delta^\nu \overline{c}_{k-\nu} \Delta^\nu \overline{c}_{k-\nu-1}.$$

Hence $H'(0) \leqslant 0$, and consequently (3.2) becomes $H'(a_1) \leqslant H'(0) \leqslant 0$. With this, theorem 3.1 is proved.

4. The inequalities in theorems 2.2, 2.3 and 3.1 can be sharpened in the following way. The variables $a_1, \ldots, a_n$ occur symmetrically in the expressions

$$b_{k-1} b_{k+1} - b_k^2 \qquad (k = 1, \ldots, n+\nu-1).$$

Therefore, for instance, instead of theorem 2.3 we have as a better result:

Theorem 4.1. *Let* ${}^i c_k$ *be the* k*-th elementary symmetric function of* $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$. *If*

$${}^i c_{k-1} - {}^i c_{k-2} > 0,$$

then

$$(c_{k-1} - c_{k-2})(c_{k+1} - c_k) - (c_k - c_{k-1})^2$$
$$\leqslant \min_{1 \leqslant i \leqslant n} \left(({}^i c_{k-1} - {}^i c_{k-2})({}^i c_{k+1} - {}^i c_k) - ({}^i c_k - {}^i c_{k-1})^2 \right).$$

5. Let $a_1, \ldots, a_\nu$ be real numbers and let ν be a positive integer. By considering the polynomial

$$(x-a_1)(x-a_2) \cdots (x-a_\nu)(c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n)$$
$$= A_0 x^{n+\nu} + A_1 x^{n+\nu-1} + \cdots + A_{n+\nu-1} x + A_{n+\nu},$$

we can obtain more general results than those proved above.

6. In the literature, and particularly in [3] and [4], we could find no inequalities of this type.

REFERENCES

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