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A PROBLEM IN GEOMETRICAL PROBABILITY*

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Introduction

If $n-1$ points are chosen at random on a given line segment of length a, we denote by m the maximal length of n intervals between consecutive points. In this paper we find the probability distribution function of m. For $n=3$ this problem is solved in many textbooks on probability, for instance in [3], p. 56, 185. For arbitrary n the probability of $m < a/2$ was found by H. H. Brazier [1], G. A. Bull [2], S. Rushton [4]. Of course, this probability is the value of the distribution function at $x=a/2$. I give two approaches to the problem: an analytic and a second one geometric. This second approach, similar to that of Brazier, is more effective.

We apply our Theorem to find the probability that n points chosen at random on a circle will lie all on some arc of length φ. This is a generalisation (for $n=2$) of a problem solved by J. G. Wendel [5]: If N points are scattered at random on the surface of the unit sphere in E^n, what is the probability that all the points lie on some hemisphere? The generalisation which we have in mind is: What is the probability that all N points lie on some portion of the sphere which has a given form?

Analytic approach

On a straight line segment AB of length a the $n-1$ points $A_1, \ldots, A_{n-1}$ are chosen at random. AB is divided into n parts by these points; we denote by x_i $(i=1, \ldots, n)$ the length of the i-th part counted from A. Let $F_n^a(x)$ be the probability distribution function of the random variable $m=\max\limits_{1\leqslant i\leqslant n} x_i$. For at least one i we have $x_i \geqslant a/n$. Hence

$$F_n^a(x)=0 \qquad (x\leqslant a/n)$$
$$=1 \qquad (x>a).$$

If $n=1$ we obtain

$$F_1^a(x)=0 \qquad (x\leqslant a)$$
$$=1 \qquad (x>a).$$

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Assume that the position of A_1 is known $AA_1 = t$. The probability that exactly $k-1$ ($k=1, \ldots, n-1$) points among A_i ($i=2, \ldots, n-1$) lie on AA_1 is

$$\binom{n-2}{k-1}\left(\frac{t}{a}\right)^{k-1}\left(\frac{a-t}{a}\right)^{n-k-1}.$$

If exactly $k-1$ points A_i ($i=2, \ldots, n-1$) lie on AA_1 the probability that the maximal part of AA_1 is not greater than x is $F_k^t(x)$; the probability that in this case the maximal part of A_1B is not greater than x is $F_{n-k}^{a-t}(x)$. Thus for $n \geqslant 2$ we obtain

(1) $$F_n^a(x) = \frac{1}{a^{n-1}} \sum_{k=1}^{n-1} \binom{n-2}{k-1} \int_0^a t^{k-1}(a-t)^{n-k-1} F_k^t(x)\, F_{n-k}^{a-t}(x)\, dt.$$

With $a^{n-1} F_n^a(x) = G_n^a(x)$ we have

$$G_n^a(x) = \sum_{k=1}^{n-1} \binom{n-2}{k-1} \int_0^a G_k^t(x)\, G_{n-k}^{a-t}(x)\, dt \qquad (n \geqslant 2).$$

Using this recurrent relation we find successively

$$\begin{aligned}
F_2^a(x) &= 0 && (u < 1/2)\\
&= 2u-1 && (1/2 < u < 1)\\
&= 1 = (2u-1) - 2(u-1) && (1 < u),
\end{aligned}$$

$$\begin{aligned}
F_3^a(x) &= 0 && (u < 1/3)\\
&= (3u-1)^2 && (1/3 < u < 1/2)\\
&= (3u-1)^2 - 3(2u-1)^2 && (1/2 < u < 1)\\
&= 1 = (3u-1)^2 - 3(2u-1)^2 + 3(u-1)^2 && (1 < u),
\end{aligned}$$

$$\begin{aligned}
F_4^a(x) &= 0 && (u < 1/4)\\
&= (4u-1)^3 && (1/4 < u < 1/3)\\
&= (4u-1)^3 - 4(3u-1)^3 && (1/3 < u < 1/2)\\
&= (4u-1)^3 - 4(3u-1)^3 + 6(2u-1)^3 && (1/2 < u < 1)\\
&= 1 = (4u-1)^3 - 4(3u-1)^3 + 6(2u-1)^3 - 4(u-1)^3 && (1 < u),
\end{aligned}$$

where $u = x/a$. Now we can guess the general formula

(2) $$\begin{aligned}
F_n^a(x) &= 0 && (x < a/n)\\
&= \sum_{r=0}^{k} (-1)^r \binom{n}{r} \left[(n-r)\frac{x}{a} - 1\right]^{n-1} && \left(\frac{a}{n-k} < x < \frac{a}{n-k-1};\ k = 0, 1, \ldots, n-1\right).
\end{aligned}$$

For $k=n-1$ we have $a<x<+\infty$ and the following identity must hold

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}[(n-r)u-1]^{n-1}=1,$$

i.e.

$$\sum_{r=1}^{n}(-1)^r\binom{n}{r}(ru-1)^{n-1}=0.$$

From this identity we get the known identities

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^k=0 \qquad (k=0,1,\ldots,n-1).$$

I could not prove (2) on the basis of (1).

Geometric approach

We shall prove the following

Theorem. *The probability distribution function $F_n^a(x)$ of the random variable m is given by* (2).

It is sufficient to prove the theorem in the case $a=1$, $u=x/a=x$. The elementary events of our experiment can be represented by points $X=(x_1,\ldots,x_n)$ in E^n. The sample space is $(n-1)$-dimensional regular simplex T defined by

$$\left.\begin{aligned} &x_i\geqslant 0 \quad (i=1,\ldots,n)\\ &x_1+\cdots+x_n=1\end{aligned}\right\} T$$

Its vertices are

$$P_i=(\underbrace{0,\ldots,0}_{i-1},1,0,\ldots,0) \qquad (i=1,\ldots,n).$$

The volume of T is

(3) $$V(T)=\sqrt{n}/(n-1)!.$$

For $u\in(-\infty,+\infty)$ we define a set S^u as the set of all points $X=(x_1,\ldots,x_n)$ such that

$$\left.\begin{aligned} &x_i\leqslant u \quad (i=1,\ldots,n)\\ &x_1+\cdots+x_n=1\end{aligned}\right\} S^u$$

The favorable cases to $m\leqslant u$ correspond to $X\in S^u\cap T$. Hence

(4) $$F_n^1(u)=\frac{V(S^u\cap T)}{V(T)}.$$

If $u<1/n$ we have $S^u\cap T=\emptyset$, $F_n^1(u)=0$. If $u\geqslant 1$ then $S^u\cap T=T$, $F_n^1(u)=1$.

Lemma 1. *If $u>1/n$ then S^u is $(n-1)$-dimensional regular simplex with vertices*

$$Q_i=(\underbrace{u,\ldots,u}_{i-1},1-(n-1)u,u,\ldots,u) \qquad (i=1,\ldots,n).$$

Proof of Lemma 1. S^u is convex since it is defined as an intersection of a finite number of closed halfspaces. Clearly, $Q_i \in S^u$ $(i=1, \ldots, n)$. The convex hull of Q_i $(i=1, \ldots, n)$ (it is a regular simplex) is contained in S^u. It remains to prove that each point $X=(x_1, \ldots, x_n) \in S^u$ has a representation $X=\lambda_1 Q_1+\cdots+\lambda_n Q_n$, with $\lambda_i \geqslant 0$ $(i=1, \ldots, n)$ and $\lambda_1+\cdots+\lambda_n=1$. We can take $\lambda_i=(u-x_i)/(nu-1)$ $(i=1, \ldots, n)$.

Lemma 2. *If* $u>1/(n-k)$ $(k=0, 1, \ldots, n-1)$ *the set* $S^u_{12\ldots k}$ *of all points* $X=(x_1, \ldots, x_n) \in S^u$ *which satisfy* $x_i \leqslant 0$ $(i=1, \ldots, k)$ *is* $(n-1)$*-dimensional regular simplex with vertices*

$$R_i=(\underbrace{0, \ldots, 0}_{i-1}, 1-(n-k)u, 0, \ldots, 0, \underbrace{u, \ldots, u}_{n-k}) \qquad (i=1, \ldots, k)$$

$$R_{k+i}=(\underbrace{0, \ldots, 0}_{k}, \underbrace{u, \ldots, u}_{i-1}, 1-(n-k-1)u, u, \ldots, u) \quad (i=1, \ldots, n-k).$$

Proof of Lemma 2. $S^u_{12\ldots k}$ is convex since it is the intersection of S^u and k closed halfspaces $x_i \leqslant 0$ $(i=1, \ldots, k)$. Clearly, $R_i \in S^u_{12\ldots k}$ $(i=1, \ldots, n)$. The convex hull of R_i $(i=1, \ldots, n)$ (it is a regular simplex) is contained in $S^u_{12\ldots k}$. It remains to prove that each point $X=(x_1, \ldots, x_n) \in S^u_{12\ldots k}$ has a representation $X=\lambda_1 R_1+\cdots+\lambda_n R_n$, with $\lambda_i \geqslant 0$ $(i=1, \ldots, n)$, λ_1+ $+\cdots+\lambda_n=1$. Since $X \in S^u_{12\ldots k} \subset S^u$ we have by Lemma 1 $X=\mu_1 Q_1+\cdots$ $\cdots+\mu_n Q_n$ with $\mu_i \geqslant 0$ $(i=1, \ldots, n)$ and $\mu_1+\cdots+\mu_n=1$. It follows that $x_i=u-\mu_i(nu-1)$ $(i=1, \ldots, n)$ and that $\mu_i(nu-1) \geqslant u$ $(i=1, \ldots, k)$. We can take

$$\lambda_i=\frac{\mu_i(nu-1)-u}{(n-k)u-1} \qquad (i=1, \ldots, k)$$

$$\lambda_i=\frac{\mu_i(nu-1)}{(n-k)u-1} \qquad (i=k+1, \ldots, n),$$

which completes the proof.

If $u>1/(n-k)$ $(k=0, 1, \ldots, n-1)$ we define analogously the regular simplex $S^u_{i_1,\ldots,i_k}$ as the set of all points $X=(x_1, \ldots, x_n) \in S^u$ which satisfy $x_{i_r} \leqslant 0$ $(r=1, \ldots, k)$. If $k=0$ it reduces to S^u. Their volumes are

$$V(S^u_{i_1,\ldots,i_k})=\frac{\sqrt{n}}{(n-1)!}((n-k)u-1)^{n-1} \qquad (k=0, 1, \ldots, n-1). \tag{5}$$

If $1/(n-k)<u<1/(n-k-1)$ $(k=0, 1, \ldots, n-1)$ then

$$V(S^u \cap T)=V(S^u)-\sum_{i_1} V(S^u_{i_1})+\sum_{i_1,i_2} V(S^u_{i_1,i_2})-\cdots + (-1)^k \sum_{i_1,\ldots,i_k} V(S^u_{i_1,\ldots,i_k}). \tag{6}$$

Let $X=(x_1, \ldots, x_n)$ be an interior point of S^u such that $x_i \neq 0$ $(i=1, \ldots, n)$. We choose a sufficiently small neighbourhood of X such that it has no points in common with coordinate hyperplanes. Assuming that X has exactly $r(\geqslant 1)$

negative coordinates $x_{i_1}, \ldots, x_{i_r}$ we get $X \in S^u_{j_1, \ldots, j_s}$ iff $j_1, \ldots, j_s$ is a subsequence of $i_1, \ldots, i_r$. From $u > 1/(n-k)$ it follows that $r \leqslant k$. Now, we can conclude that the volume of a chosen neighbourhood of X is counted in the right hand side of (6) N_X times

$$N_X = 1 - \binom{r}{1} + \binom{r}{2} - \cdots + (-1)^r \binom{r}{r} = 0.$$

If $r = 0$, i.e. $X \in S^u \cap T$ then the volume of a chosen neighbourhood of X is taken into account only in $V(S^u)$. This proves (6), and (2) is implied by (3), (4), (5) and (6).

Division of a circle

Let n points A_i $(i = 1, \ldots, n)$ be chosen at random on circumference of the unit circle. We shall find the probability $\Phi_n(\varphi)$ that all these n points lie on some arc of length φ.

We denote by θ_i $(i = 1, \ldots, n)$ the lengths of n arcs so obtained. The probability which we seek is equal to the probability that for at least one i we have $\theta_i \geqslant 2\pi - \varphi$, i.e.

$$\Phi_n(\varphi) = 1 - F_n^{2\pi}(2\pi - \varphi),$$

where $F_n^a(x)$ is given by (1). Hence,

$$\begin{aligned} \Phi_n(\varphi) &= 0 && (\varphi < 0) \\ &= \sum_{s=1}^{m-1} (-1)^{n+s} \binom{n}{s} \left(s - 1 - \frac{s\varphi}{2\pi}\right)^{n-1} && \left(2\pi \frac{m-2}{m-1} < \varphi < 2\pi \frac{m-1}{m},\ m = 2, \ldots, n\right) \\ &= 1 && \left(\varphi > 2\pi \frac{n-1}{n}\right). \end{aligned}$$

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