PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 164 (1965)

A REMARK ON A NOTE BY S. FEMPL "SOME INEQUALITIES INVOLVING ELLIPTIC FUNCTIONS"*

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P. L. Duren [1] has derived the following inequalities for Jacobi elliptic functions:

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u+\operatorname{cn} u)^2} \leqslant \frac{u}{K(1-k^2)}, \qquad 0 \leqslant u \leqslant K; \tag{1}$$

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u+k \operatorname{cn} u)^2} \geqslant \frac{2u}{\pi(1+k)^2}, \qquad 0 \leqslant u \leqslant K, \tag{2}$$

where

$$K=\int_0^{\pi/2}(1-k^2 \sin^2 \theta)^{-1/2}\, d\theta .$$

In a note by S. Fempl [2] the following inequalities appear:

$$\frac{\operatorname{cn} u}{\operatorname{dn} u+\operatorname{cn} u} \leqslant \frac{K^2-u^2}{2K}; \tag{3}$$

$$\frac{\operatorname{cn} u}{\operatorname{dn} u+k \operatorname{cn} u} \geqslant \frac{1-k}{1+k} \frac{K^2-u^2}{\pi}; \tag{4}$$

$$\frac{\operatorname{sn} u}{u} \leqslant \frac{2K-u}{2K-u(2K-u)}. \tag{5}$$

It is not explicitly stated for what values of u these inequalities hold. It may be seen that these are $0 \leqslant u \leqslant K$ for (3) and (4), and $0<u \leqslant K$, $2K-u(2K-u)>0$ for (5). The inequalities (3), (4), (5) have been obtained from (1) and (2).

We shall show that inequalities (3) and (5) are simple consequences of the well known inequality

$$\frac{1}{K} \leqslant \frac{\operatorname{sn} u}{u} \leqslant 1, \qquad 0<u \leqslant K. \tag{6}$$

The second inequality in (6) is sharper than (5) since

$$\frac{2K-u}{2K-u(2K-u)} \geqslant 1.$$

for all u satisfying $0<u \leqslant K$, $2K-u(2K-u)>0$.

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* Presented december 1, 1965 by D. S. Mitrinović.

Putting $K-u$ instead of u in (6), we obtain

$$\frac{K-u}{K} \leqslant \frac{\operatorname{cn} u}{\operatorname{dn} u} \leqslant K-u, \qquad 0 \leqslant u \leqslant K,$$

which implies

$$\frac{K-u}{2K-u} \leqslant \frac{\operatorname{cn} u}{\operatorname{dn} u+\operatorname{cn} u} \leqslant \frac{K-u}{K-u+1}, \qquad 0 \leqslant u \leqslant K, \tag{7}$$

$$\frac{K-u}{K+k(K-u)} \leqslant \frac{\operatorname{cn} u}{\operatorname{dn} u+k \operatorname{cn} u} \leqslant \frac{K-u}{1+k(K-u)}, \qquad 0 \leqslant u \leqslant K. \tag{8}$$

The second inequality in (7) is sharper than (3) since

$$\frac{K-u}{K-u+1} \leqslant \frac{K^2-u^2}{2K}, \qquad 0 \leqslant u \leqslant K.$$

It may be shown that

$$\frac{K-u}{K+k(K-u)} \geqslant \frac{1-k}{1+k} \frac{K^2-u^2}{\pi}, \qquad 0 \leqslant u \leqslant K$$

if $k \geqslant \sqrt{\frac{\pi}{2}}-1 \approx 0{,}25$. For small values of k these functions are not comparable on the interval $0 \leqslant u \leqslant K$. Hence, if $\sqrt{\frac{\pi}{2}}-1 \leqslant k \leqslant 1$ the first inequality in (8) is sharper than (4).

REFERENCES

[1] P. L. Duren, *Two inequalities involving elliptic functions*, The American Mathematical Monthly, **70** (1963), 650—651.

[2] S. Fempl, *Some inequalities involving elliptic functions*, The American Mathematical Monthly, **72** (1965), 150—152.