PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 163 (1966)

SOME FORMULAS FOR THE GENERALIZED LAGUERRE POLYNOMIALS*

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1. In a recent paper [1], we have defined the generalized Laguerre polynomials $T_{kn}^{(\alpha)}(x, p)$ by the Rodrigues' formula

$$(1.1)\qquad T_{kn}^{(\alpha)}(x, p)=\frac{1}{n!}x^{-\alpha}e^{px^k}\,\mathrm{D}^n(x^{\alpha+n}e^{-px^k}),$$

where k is a natural number. The polynomial $T_{kn}^{(\alpha)}(x,p)$ is of exactly degree kn. In [1, p. 186] we notice that

$$(1.2)\qquad \binom{m+n}{m}T_{k(m+n)}^{(\alpha)}(x,p)=\sum_{r=0}^{\min(m,kn)}\frac{x^r}{r!}T_{(m-r)}^{(\alpha+n+r)}(x, p)\,\mathrm{D}^r\,T_{kn}^{(\alpha)}(x,p).$$

In an earlier work [2], we defined the polynomials $T_{kn}^{(\alpha)}(x)$ by the Rodrigues' formula

$$(1.3)\qquad T_{kn}^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha}e^{x^k}\,\mathrm{D}^n(x^{\alpha+n}e^{-x^k}).$$

Thus when $p=1$, $T_{kn}^{(\alpha)}(x,p)\equiv T_{kn}^{(\alpha)}(x)$. In fact, when $p=1$, we have

$$(1.4)\qquad \binom{m+n}{m}T_{k(m+n)}^{(\alpha)}(x)=\sum_{r=0}^{\min(m,kn)}\frac{x^r}{r!}T_{k(m-r)}^{(\alpha+n+r)}(x)\,\mathrm{D}^r\,T_{kn}^{(\alpha)}(x).$$

Subsequently in a paper [3] we have found operational derivation of the following results for $T_{kn}^{(\alpha)}(x,p)$:

$$(1.5)\qquad \sum_{n=0}^{\infty}T_{kn}^{(\alpha)}(x,p)\,t^n=(1-t)^{-\alpha-1}e^{px^k\{1-(1-t)^{-k}\}}$$

$$(1.6)\qquad \sum_{n=0}^{\infty}T_{kn}^{(\alpha-n)}(x,p)\,t^n=(1+t)^{\alpha}e^{px^k\{1-(1+t)^{k}\}}$$

$$(1.7)\qquad \sum_{m=0}^{\infty}\binom{m+n}{m}T_{k(m+n)}^{(\alpha)}(x,p)\,t^m=(1-t)^{-\alpha-n-1}e^{px^k\{1-(1-t)^{-k}\}}\,T_{kn}^{(\alpha)}\left(\frac{x}{1-t}, p\right)$$

$$(1.8)\qquad \sum_{n=0}^{\infty}\binom{m+n}{m}T_{k(m+n)}^{(\alpha-n)}(x, p)\,t^n=(1+t)^{\alpha}e^{px^k\{1-(1-t)^{k}\}}\,T_{km}^{(\alpha)}(x(1+t), p)$$

Here we shall show that the results (1.7) and (1.8) can be obtained by using (1.2) and the analogous formula (2.3), derived in the next section.

* Presented december 1, 1965 by D. S. Mitrinović.

2. First we shall derive (1.7) from (1.2). It follows from (1.2) that

$$\sum_{m=0}^{\infty}\binom{m+n}{n} T_{k(m+n)}^{(\alpha)}(x,p)\,t^m$$

$$=\sum_{r=0}^{kn}\frac{(xt)^r}{r!}\,\mathrm{D}^r T_{kn}^{(\alpha)}(x,p)\sum_{m=0}^{\infty} T_{km}^{(\alpha+n+r)}(x,p)\,t^m$$

$$=\sum_{r=0}^{kn}\frac{(xt)^r}{r!}\,\mathrm{D}^r T_{kn}^{(\alpha)}(x,p)\cdot(1-t)^{-\alpha-n-r-1}\,e^{px^k\{1-(1-t)^{-k}\}}$$

$$=(1-t)^{-\alpha-n-1}\,e^{px^k\{1-(1-t)^{-k}\}}\sum_{r=0}^{kn}\left(\frac{xt}{1-t}\right)^r\frac{1}{r!}\,\mathrm{D}^r T_{kn}^{(\alpha)}(x,p).$$

Now

$$T_{kn}^{(\alpha)}\left(\frac{x}{1-t},\,p\right)=T_{kn}^{(\alpha)}\left(x+\frac{xt}{1-t},\,p\right)=\sum_{r=0}^{kn}\left(\frac{xt}{1-t}\right)^r\frac{1}{r!}\,\mathrm{D}^r T_{kn}^{(\alpha)}(x,p).$$

Thus we obtain

$$\sum_{m=0}^{\infty}\binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x,p)\,t^m=(1-t)^{-\alpha-n-1}\,e^{px^k\{1-(1-t)^{-k}\}}\,T_{kn}^{(\alpha)}\left(\frac{x}{1-t},\,p\right), \tag{2.1}$$

which is (1.7). Conversely (2.1) implies (1.2).

Next to derive (1.8) we require an analogous formula of (1.2). It may be noted here that (1.2) is equivalent to the following formula:

$$\binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x,p)=\sum_{r=0}^{\min(n,\,km)}\frac{x^r}{r!}\,T_{k(n-r)}^{(\alpha+m+r)}(x,p)\,\mathrm{D}^r T_{km}^{(\alpha)}(x,p). \tag{2.2}$$

Now we shall prove that

$$\binom{m+n}{m} T_{k(m+n)}^{(\alpha-n)}(x,p)=\sum_{r=0}^{\min(n,\,km)}\frac{x^r}{r!}\,T_{k(n-r)}^{(\alpha-n+r)}(x,p)\,\mathrm{D}^r T_{km}^{(\alpha)}(x,p). \tag{2.3}$$

In proving (2.3) we make use of the result [1, p. 183]

$$\prod_{j=1}^{n}(x\mathrm{D}-pk\,x^k+\alpha+j)\,Y=n!\,\sum_{r=0}^{n}\frac{x^r}{r!}\,T_{k(n-r)}^{(\alpha+r)}(x,p)\,\mathrm{D}^r\,Y. \tag{2.4}$$

Thus we obtain

$$(m+n)!\;T_{k(m+n)}^{(\alpha-n)}(x,p)=\prod_{j=1}^{m+n}(x\mathrm{D}-pk\,x^k+\alpha-n+j)\cdot 1$$

$$=\prod_{j=1}^{n}(x\mathrm{D}-pk\,x^k+\alpha-n+j)\prod_{i=1}^{m}(x\mathrm{D}-pk\,x^k+\alpha+i)\cdot 1$$

$$=m!\,\prod_{j=1}^{n}(x\mathrm{D}-pk\,x^k+\alpha-n+j)\,T_{km}^{(\alpha)}(x,p)$$

$$=m!\;n!\,\sum_{r=0}^{n}\frac{x^r}{r!}\,T_{k(n-r)}^{(\alpha-n+r)}(x,p)\,\mathrm{D}^r T_{km}^{(\alpha)}(x,p)$$

which implies that

$$\binom{m+n}{m} T_{k(m+n)}^{(\alpha-n)}(x, p) = \sum_{r=0}^{\min(n, km)} \frac{x^r}{r!} T_{k(n-r)}^{(\alpha-n+r)} \mathrm{D}^r T_{km}^{(\alpha)}(x, p).$$

Now we shall deduce (1.8) by using the formula (2.3).

We have

$$\sum_{n=0}^{\infty} \binom{m+n}{m} T_{k(m+n)}^{(\alpha-n)}(x, p)\, t^n = \sum_{r=0}^{km} \frac{(xt)^r}{r!} \mathrm{D}^r T_{km}^{(\alpha)}(x, p) \sum_{n=0}^{\infty} T_{kn}^{(\alpha-n)}(x, p)\, t^n$$

$$= \sum_{r=0}^{km} \frac{(xt)^r}{r!} \mathrm{D}^r T_{km}^{(\alpha)}(x, p) \cdot (1+t)^\alpha e^{px^k\{1-(1+t)^k\}}.$$

But

$$T_{km}^{(\alpha)}(x(1+t), p) = T_{km}^{(\alpha)}(x+xt, p) = \sum_{r=0}^{km} \frac{(xt)^r}{r!} \mathrm{D}^r T_{km}^{(\alpha)}(x, p).$$

Thus it follows that

$$(2.5) \qquad \sum_{n=0}^{\infty} \binom{m+n}{m} T_{k(m+n)}^{(\alpha-n)}(x, p)\, t^n = (1+t)^\alpha e^{px^k\{1-(1+t)^k\}}\, T_{km}^{(\alpha)}(x(1+t), p),$$

which is (1.8). Conversely (2.5) implies (2.3).

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