PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 152 (1965)

A SPECIAL CYCLIC FUNCTIONAL EQUATION*

Dragomir Ž. Djoković

1. Introduction. — Let $F_i(x,y)$, $i=1, \ldots, m+n$ be unknown real valued functions of two real variables, which satisfy

$$\sum_{i=1}^{m+n} F_i(x_i+x_{i+1}+\cdots+x_{i+m-1},\ x_{i+m}+x_{i+m+1}+\cdots+x_{i+m+n-1})=0, \tag{1}$$

where x_i, $i=1, \ldots, m+n$ are independent variables and $x_{i+m+n}=x_i$, $i=1, 2, \ldots$ In [2] the general continuous solution of (1) is found. But, the argument given there is valid only if the greatest common divisor of m and n is $(m,n)=1$. The general case is not solved, which will be done in this paper.

2. A Correction. — Let us restate the theorem from [2] in the correct form.

Theorem A. *The general continuous solution of* (1) *if* $(m,n)=1$, $m+n>2$, $m>0$, $n>0$, *is*

$$F_i(x,y)=(nx-my)\,f(x+y)+g_i(x+y), \qquad i=1, \ldots, m+n; \tag{2}$$

$$\sum_{i=1}^{m+n} g_i(x)=0, \tag{3}$$

where $f(x)$ *and* $g_i(x)$, $i=1, \ldots, m+n-1$ *are arbitrary continuous functions.*

The proof of Theorem A which is given in [2] is valid since $(m,n)=1$ implies that the variables

$$t_i=x_i+x_{i+1}+\cdots+x_{i+m-1}-\frac{ms}{m+n}, \qquad i=1, \ldots, m+n-1;$$

and

$$s=x_1+x_2+\cdots+x_{m+n}$$

are independent. In order to prove their independence we can use instead of t_i, $i=1, \ldots, m+n-1$ and s the variables

$$T_i=x_i+x_{i+1}+\cdots+x_{i+m-1}, \qquad i=1, \ldots, m+n \tag{4}$$

since

$$t_i=T_i-\frac{1}{m+n}\sum_{i=1}^{m+n} T_i, \qquad i=1, \ldots, m+n-1; \qquad s=\frac{1}{m}\sum_{i=1}^{m+n} T_i$$

* Presented March 20, 1965 by D. S. Mitrinović.

and

$$T_i = t_i + \frac{ms}{m+n}, \qquad i = 1, \ldots, m+n-1;$$

$$T_{m+n} = -(t_1 + t_2 + \cdots + t_{m+n-1}) + \frac{ms}{m+n}.$$

The determinant D of the system (4) of linear forms is a circulant whose first row is $\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, 0, \ldots, 0}_{n}$.

Hence,

$$D = \alpha(1)\,\alpha(\varepsilon)\,\alpha(\varepsilon^2) \cdots \alpha(\varepsilon^{m+n-1})$$

where

$$\alpha(x) = 1 + x + x^2 + \cdots + x^{m-1} = \frac{x^m - 1}{x-1}, \qquad \varepsilon = \exp\frac{2\pi i}{m+n}.$$

D will vanish if and only if for some $k = 1, \ldots, m+n-1$ we have $\varepsilon^{km} = 1$, i.e. if and only if $(m, n) > 1$. Hence, if $(m, n) = 1$ the variables T_i, $i = 1, \ldots, m+n$ are independent and so are the variables t_i, $i = 1, \ldots, m+n-1$ and s.

A minor completion of the proof in [2] furnishes the following

Theorem B. *The most general solution of* (1) *if* $(m, n) = 1$, $m+n > 2$, $m > 0$, $n > 0$, *is*

$$F_i(x, y) = \varphi(nx - my)\, f(x+y) + g_i(x+y), \qquad i = 1, \ldots, m+n; \tag{5}$$

$$\sum_{i=1}^{m+n} g_i(x) = 0, \tag{6}$$

where $f(x)$ *and* $g_i(x)$, $i = 1, \ldots, m+n-1$ *are arbitrary functions and* $\varphi(x)$ *is the general solution of the Cauchy functional equation*

$$\varphi(x+y) = \varphi(x) + \varphi(y). \tag{7}$$

Remark. We can write the solution also in the following form

$$F_i(x, y) = \varphi(x)\, f(x+y) + g_i(x+y), \qquad i = 1, \ldots, m+n;$$

$$\sum_{i=1}^{m+n} g_i(x) = -m\,\varphi(x)\, f(x).$$

Corollary 1. *The most general solution of the functional equation*

$$\sum_{i=1}^{m+n} G_i(x_i + x_{i+1} + \cdots + x_{i+m-1},\ x_1 + x_2 + \cdots + x_{m+n}) = 0 \tag{8}$$

if $(m, n) = 1$, $m+n > 2$, $m > 0$, $n > 0$, *is*

$$G_i(x, y) = \varphi(x)\, f(y) + g_i(y), \qquad i = 1, \ldots, m+n; \tag{9}$$

$$\sum_{i=1}^{m+n} g_i(y) = -m\,\varphi(y)\, f(y), \tag{10}$$

where $f(y)$ *and* $g_i(y)$, $i = 1, \ldots, m+n-1$ *are arbitrary functions and* $\varphi(x)$ *is the general solution of* (7).

Corollary 2. *The most general solution of the functional equation*

$$(11)\qquad \sum_{i=1}^{m+n} H_i(x_i+x_{i+1}+\cdots+x_{i+m-1}, x_1+x_2+\cdots+x_{m+n}) = H(x_1+x_2+\cdots+x_{m+n})$$

if $(m, n)=1,\ m+n>2,\ m>0,\ n>0,$ *is*

$$(12)\qquad H_i(x, y)=\varphi(x) f(y)+g_i(y), \qquad i=1, \ldots, m+n;$$

$$(13)\qquad \sum_{i=1}^{m+n} g_i(y)=H(y)-m\varphi(y) f(y),$$

where $f(y)$ *and* $g_i(y),\ i=1, \ldots, m+n-1$ *are arbitrary functions and* $\varphi(x)$ *is the general solution of* (7).

Proofs. Corollary 1 follows by putting $G_i(x, y)=F_i(x, y-x)$. Corollary 2 follows from Corollary 1 by introducing $G_i(x, y)=H_i(x, y)-H(y)/(m+n)$.

3. Main Results. — Firstly we prove

Theorem 1. *The most general solution of the functional equation* (8), *if* $(m, n)=d>1,\ m/d=\mu,\ n/d=\nu,\ \mu+\nu>2,$ *is*

$$(14)\qquad G_{id+j}(x, y)=\varphi^j(x) f^j(y)+g_i^j(y), \quad i=0, 1, \ldots, \mu+\nu-1,\ j=1, \ldots, d;$$

$$(15)\qquad \sum_{j=1}^{d} H^j(y)=0;$$

$$(16)\qquad \sum_{i=0}^{\mu+\nu-1} g_i^j(y)=H^j(y)-\mu\varphi^j(y) f^j(y), \qquad j=1, \ldots, d,$$

where $f^j(y),\ j=1, \ldots, d;\ H^j(y),\ j=1, \ldots, d-1;\ g_i^j(y),\ j=1, \ldots, d,$ $i=0, 1, \ldots, \mu+\nu-2$ *are arbitrary functions and* $\varphi^j(y),\ j=1, \ldots, d$ *are the general solutions of* (7).

Proof. Let us introduce the new variables

$$(17)\qquad y_i=x_i+x_{i+1}+\cdots+x_{i+d-1},\ i=1, \ldots, m+n \quad (y_{i+m+n}=y_i)$$

and

$$(18)\qquad z=x_1+x_2+\cdots+x_{m+n}.$$

They are not independent since

$$(19)\qquad \sum_{i=0}^{\mu+\nu-1} y_{id+j}=z, \quad j=1, \ldots, d.$$

The variables $y_i,\ i=1, \ldots, m+n-d,$ and z are independent since it is easy to see that the rank of the matrix of linear forms determining them is $m+n-d+1$. In the sequel we shall use all variables (17) and (18) but we must have allways in mind that (19) holds. The equation (8) is now

$$\sum_{i=1}^{m+n} G_i(y_i+y_{i+d}+\cdots+y_{i+(\mu-1)d},\ z)=0.$$

It can be rewritten in the form

$$\sum_{j=1}^{d}\sum_{i=0}^{\mu+\nu-1} G_{id+j}(y_{id+j}+y_{(i+1)d+j}+\cdots+y_{(i+\mu-1)d+j}, z)=0 .$$

If we set here

$$y_{id+j}=0,\ i=0, 1, \ldots, \mu+\nu-2,\ j=1, 2, \ldots, r-1,\ r+1, \ldots, d;$$

$$y_{(\mu+\nu-1)d+j}=z,\ j=1, 2, \ldots, r-1,\ r+1, \ldots, d,$$

we get

$$\sum_{i=0}^{\mu+\nu-1} G_{id+r}(y_{id+r}+y_{(i+1)d+r}+\cdots+y_{(i+\mu-1)d+r}, z)=H^r(z),\ r=1, \ldots, d,$$

and

$$\sum_{r=1}^{d} H^r(z)=0 .$$

Remembering that $(\mu, \nu)=1$ and by using Corollary 2 of Theorem B we get

$$G_{id+r}(x, y)=\varphi^r(x)f^r(y)+g_i^r(y),\ i=0, 1, \ldots, \mu+\nu-1,\ r=1, \ldots, d;$$

$$\sum_{i=0}^{\mu+\nu-1} g_i^r(y)=H^r(y)-\mu\varphi^r(y)f^r(y),\ r=1, \ldots, d,$$

where $f^r(y)$, $r=1, \ldots, d$; $g_i^r(y)$, $i=0, 1, \ldots, \mu+\nu-2$, $r=1, \ldots, d$ are arbitrary functions and $\varphi^r(y)$, $r=1, \ldots, d$ are the general solutions of (7).

Conversely, any system of functions $G_i(x, y)$, $i=1, \ldots, m+n$ defined by (14), (15), (16) satisfies (8). This completes the proof.

Corollary 1. *The most general solution of* (1) *if* $(m, n)=d>1$, $m/d=\mu$, $n/d=\nu$, $\mu+\nu>2$, *is*

(20) $$F_{id+j}(x, y)=\varphi^j(x)f^j(x+y)+g_i^j(x+y),\ i=0, 1, \ldots, \mu+\nu-1,\ j=1, \ldots, d;$$

(21) $$\sum_{j=1}^{d} H^j(y)=0;$$

(22) $$\sum_{i=0}^{\mu+\nu-1} g_i^j(y)=H^j(y)-\mu\varphi^j(y)f^j(y),\ j=1, \ldots, d,$$

where $f^j(y), j=1, \ldots, d$; $H^j(y)$, $j=1, \ldots, d-1$; $g_i^j(y)$, $j=1, \ldots, d$, $i=0, 1, \ldots, \mu+\nu-2$ *are arbitrary functions and* $\varphi^j(x)$, $j=1, \ldots, d$ *are the general solutions of* (7).

Proof. Put $F_i(x, y)=G_i(x, x+y)$.

The exceptional case $m=n$ is solved by

Theorem 2. *The most general solution of* (8) *if* $m=n$ *is*

(23) $G_i(x, y)$, $i=1, \ldots, m$, *are arbitrary;*

(24) $$G_{m+i}(x, y)=H_i(y)-G_i(y-x, y),\quad i=1, \ldots, m;$$

(25) $$\sum_{i=1}^{m} H_i(y)=0,$$

where $H_i(y)$, $i=1, \ldots, m-1$ *are also arbitrary functions.*

Proof. Putting in (14)

$$y_i = x_i + x_{i+1} + \cdots + x_{i+m-1}, \quad i = 1, \ldots, m; \quad y = x_1 + x_2 + \cdots + x_{2m}$$

we get

$$\sum_{i=1}^{m} [G_i(y_i, y) + G_{m+i}(y - y_i, y)] = 0.$$

It follows that

$$G_i(y_i, y) + G_{m+i}(y - y_i, y) = H_i(y), \quad i = 1, \ldots, m; \quad \sum_{i=1}^{m} H_i(y) = 0.$$

This proves the theorem.

Corollary 1. *The most general solution of* (1) *if* $m = n$ *is*

(26) $F_i(x, y), \quad i = 1, \ldots, m$ *are arbitrary;*

(27) $F_{m+i}(x, y) = H_i(x+y) - F_i(y, x), \quad i = 1, \ldots, m;$

(28) $$\sum_{i=1}^{m} H_i(x) = 0,$$

where $H_i(x)$, $i = 1, \ldots, m-1$ *are arbitrary functions.*

Proof. Put $F_i(x, y) = G_i(x, x+y)$.

Remark. The general continuous solutions are given by the same formulae with the additional condition that all the arbitrary functions are continuous and that $\varphi(x)$ and $\varphi^r(x)$ should be replaced by x.

4. Some Applications. — It is clear that the functional equation

(29) $$\sum_{i=1}^{m+n} a_i f(x_i + x_{i+1} + \cdots + x_{i+m-1}, x_{i+m} + x_{i+m+1} + \cdots + x_{i+m+n-1}) = 0$$
$$(\text{some } a_i \neq 0),$$

is a special case of the equation (1). The last equation with a minor unessential modification was considered by Jong [1]. The results of Jong are easy consequences of our theorems. We shall show this in the case $m = n$. By Corollary 1 of Theorem 2 we have

(30) $a_i f(x, y) = F_i(x, y), \quad i = 1, \ldots, m;$

(31) $a_{m+i} f(x, y) = H_i(x+y) - F_i(y, x), \quad i = 1, \ldots, m;$

(32) $$\sum_{i=1}^{m} H_i(x) = 0.$$

From (30) and (31) we get

(33) $a_{m+i} f(x, y) + a_i f(y, x) = H_i(x+y), \quad i = 1, \ldots, m.$

I *Case:* $a_i = a_{m+i}$, $i = 1, \ldots, m$.

Summing (33) over i and taking (32) into account we find

$$S(f(x, y) + f(y, x)) = 0 \quad \left(S = \sum_{i=1}^{m} a_i\right).$$

If $S\neq 0$ then $f(x, y)+f(y, x)=0$, i. e.

$$f(x, y)=A(x, y)-A(y, x), \tag{34}$$

where $A(x, y)$ is an arbitrary function. If $S=0$ we get from (33) that $f(x, y)+f(y, x)=2B(x+y)$, i. e.

$$f(x, y)=B(x+y)+C(x, y)-C(y, x), \tag{35}$$

where $B(x)$ and $C(x, y)$ are arbitrary functions. The general solution is given by (34) and (35).

II *Case:* $-a_i=a_{m+i}$, $i=1, \ldots, m$.

From (33) interchanging x and y we get $H_i(x+y)=0$. Hence, $f(x, y)=$ $=f(y, x)$, i. e.

$$f(x, y)=A(x, y)+A(y, x), \tag{36}$$

where $A(x, y)$ is an arbitrary function.

III *Case:* $a_i\neq a_{m+i}$ and $-a_j\neq a_{m+j}$ for some i and j.

From (33) we have

$$a_{m+i}f(x, y)+a_i f(y, x)=H_i(x+y),$$

$$a_{m+i}f(y, x)+a_i f(x, y)=H_i(x+y),$$

$$(a_{m+i}-a_i)(f(x, y)-f(y, x))=0,$$

$$f(x, y)=f(y, x).$$

If $\sum_{i=1}^{2m} a_i\neq 0$, summing (33) over i and using $f(x, y)=f(y, x)$, we get

$$f(x, y)=0. \tag{37}$$

If $\sum_{i=1}^{2m} a_i=0$ then $f(x, y)=f(y, x)$ and (33) gives

$$f(x, y)=B(x+y), \tag{38}$$

where $B(x)$ is an arbitrary function. The general solution of (29) in this case is given by (37) and (38).

BIBLIOGRAPHY

[1] J. K. Jong: *A class of linear functional equations*, These Publications, № 151.

[2] S. Prešić — D. Ž. Djoković: *Sur une équation fonctionnelle*, Bulletin de la Société des mathématiciens et physiciens de la R. P. de Serbie, Beograd, **13** (1961), 140—152.