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A CLASS OF LINEAR FUNCTIONAL EQUATIONS*

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In this paper we shall consider the functional equation

$$(1)\qquad \sum_{i=1}^{n+m} a_i f(x_i + x_{i+1} + \cdots + x_{i+n-1},\ x_{i+n} + \cdots + x_{i+m+n-1})$$
$$+ a' f(0,\ x_1 + x_2 + \cdots + x_{m+n}) + a'' f(x_1 + x_2 + \cdots + x_{m+n},\ 0) = 0$$

where a_i, a' and a'' are arbitrary real numbers, all a_i are not zero; n and m are two arbitrary positive integers; x_i are independent variables and

$$x_{n+m+k} = x_k \quad (k = 1,\ 2,\ \ldots,\ n+m-1).$$

The certain particular cases of (1) are solved [1], [2]. We shall give a method for solving (1) in the general case.[1]

Let $f(x,\ y) = F(x,\ x+y)$ and by putting $x_1 + x_2 + \cdots + x_{n+m} = t$ in (1), we have

$$(2)\qquad \sum_{i=1}^{n+m} a_i F(x_i + x_{i+1} + \cdots + x_{i+n-1},\ t) + a' F(0,\ t) + a'' F(t,\ t) = 0.$$

There are three cases

(i) $n + m = 2n$ i. e. $n = m$;

(ii) $n + m = Mn$ $(M > 2)$;

(iii) $n + m = Mn + r$ $(1 \leqslant r \leqslant n-1)$.

(i) Let $n = m$. Let in (2)

$$(3)\qquad x_i + x_{i+1} + \cdots + x_{i+n-1} = y_i \quad (i = 1,\ 2,\ \ldots,\ n),$$
$$x_1 + x_2 + \cdots + x_{2n} = t$$

then

$$(4)\qquad x_{n+i} + x_{n+i+1} + \cdots + x_{2n+i-1} = t - y_i \quad (i = 1,\ 2,\ \ldots,\ n).$$

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[1] *Remarks of the Redaction:* I. The author in case (iii) gives only the general continuous solution of a particular equation, not for the general case.

II. The two terms $a' f(0,\ x_1 + x_2 + \cdots + x_{n+m})$ and $a'' f(x_1 + x_2 + \cdots + x_{n+m},\ 0)$ can be omitted from (1) without any loss of generality. This would simplify many of the later statements.

The first n equations of (3) are mutually independent and the last equation of (3) has an independent variable x_{2n} which does not appear in the first n equations of (3). Hence $y_1, y_2, \ldots, y_n$ and t are independent variables.

It we set (3) and (4) into (2), we have

$$\sum_{i=1}^{n} [a_i F(y_i, t) + a_{n+i} F(t-y_i, t)] + a' F(0, t) + a'' F(t, t) = 0. \tag{5}$$

Suppose that a certain $a_{i_0} \neq 0$ in (5). By fixing the variables

$$y_1, y_2, \ldots, y_{i_0-1}, \quad y_{i_0+1}, \ldots, y_n$$

and denoting the sum of all summands of (5), except $i = i_0$, by $-2a_{i_0}\,\alpha(t)$, then we have

$$a_{i_0} F(y_{i_0}, t) + a_{n+i_0} F(t-y_{i_0}, t) = 2a_{i_0}\,\alpha(t).$$

We have to consider three cases.

1° If $a_{n+i_0} = a_{i_0}$, then $F(y, t) = A(y, t) - A(t-y, t) + \alpha(t)$. If we put this into (5), we have

$$\sum_{i=1}^{n} (a_i - a_{n+i})[A(y_i, t) - A(t-y_i, t)] + \alpha(t) \sum_{i=1}^{n} (a_i + a_{n+i})$$
$$+ (a' + a'')\,\alpha(t) + (a' - a'')[A(0, t) - A(t, t)] = 0.$$

Therefore, if $a_{n+i} = a_i$ $(i = 1, 2, \ldots, n)$ and

1° (a) $\quad a' = a'',\ a' + \sum_{i=1}^{n} a_i = 0$ then $F(y, t) = A(y, t) - A(t-y, t) + \alpha(t)$;

1° (b) $\quad a' + a'' + 2\sum_{i=1}^{n} a_i \neq 0$ then

$$F(y, t) = A(y, t) - A(t-y, t) + \frac{(a'' - a')[A(0, t) - A(t, t)]}{a' + a'' + 2\sum_{i=1}^{n} a_i};$$

1° (c) $\quad a' \neq a'',\quad a' + a'' + 2\sum_{i=1}^{n} a_i = 0$, then $F(y, t) = A(y, t) - A(t-y, t)$,

where $A(t, t) = A(0, t)$.

2° If $a_{n+i_0} = -a_{i_0}$, it is clear that we also have $\alpha(t) = 0$, then

$$F(y, t) = A(y, t) + A(t-y, t).$$

If we put this into (5), we have

$$\sum_{i=1}^{n} (a_i + a_{n+i})[A(y_i, t) + A(t-y_i, t)] + (a' + a'')[A(0, t) + A(t, t)] = 0.$$

Therefore, if $a_{n+i} = -a_i \quad (i = 1, 2, \ldots, n)$ and

2° (a) $\quad a' = -a''$ then $F(y, t) = A(y, t) + A(t-y, t)$;

2° (b) $a' \neq -a''$ then $F(y, t) = A(y, t) + A(t-y, t)$ where $A(t, t) = -A(0, t)$.

3° If i and j exist such that $a_{n+i} \neq a_i$, $a_{n+j} \neq -a_j$, and $a' + a'' + 2\sum_{i=1}^{n} a_i = 0$, then (5) has only the solution $F(y, t) = \alpha(t)$.

By putting above results into $f(x, y) = F(x, x+y)$ we have the

Theorem 1. *The general solution of the functional equation* (1) *for* $n = m$ *has the following forms:*

1. $f(x, y) = A(x, x+y) - A(y, x+y) + \alpha(x+y)$ *if* $a_{n+i} = a_i$ $(i = 1, 2, \ldots, n)$ *and* $a' = a''$, $a' + \sum_{i=1}^{n} a_i = 0$;

2. $f(x, y) = A(x, x+y) - A(y, x+y) + \dfrac{(a''-a')[A(0, t) - A(t, t)]}{a' + a'' + 2\sum_{i=1}^{n} a_i}$ *if* $a_{n+i} = a_i$ $(i = 1, 2, \ldots, n)$ *and* $a' + a'' + 2\sum_{i=1}^{n} a_i \neq 0$;

3. $f(x, y) = A(x, x+y) - A(y, x+y)$ *where* $A(t, t) = A(0, t)$, *if* $a_{n+i} = -a_i$ $(i = 1, 2, \ldots n)$ *and* $a' \neq -a''$;

4. $f(x, y) = A(x, x+y) + A(y, x+y)$ *if* $a_{n+i} = -a_i$ $(i = 1, 2, \ldots, n)$ *and* $a' = -a''$;

5. $f(x, y) = A(x, x+y) + A(y, x+y)$ *where* $A(t, t) = -A(0, t)$ *if* $a_{n+i} = -a^i$ $(i = 1, 2, \ldots, n)$ *and* $a' \neq -a''$;

6. $f(x, y) = \alpha(x+y)$ *if* i *and* j *exist such that* $a_{n+i} \neq a_i$, $a_{n+j} \neq -a_j$, *and* $\sum_{i=1}^{n} (a_i + a_{n+i}) = 0$;

7. $f(x, y) = 0$, *in other cases;*

where $A(x, y)$ *and* $\alpha(t)$ *are arbitrary functions.*

(ii) Now we take $n + m = Mn$ $(M > 2)$. Let in (2)

(6) $$x_i + x_{i+1} + \cdots + x_{i+n-1} = y_i \quad (i = 1, 2, \ldots, Mn)$$
$$x_1 + x_2 + \cdots + x_{Mn} = t.$$

We have

$$t = (x_i + x_{i+1} + \cdots + x_{i+n-1}) + (x_{i+n} + x_{i+n+1} + \cdots + x_{i+2n-1}) + \cdots$$
$$+ (x_{i+(M-2)n} + \cdots + x_{i+(M-1)n-1}) + (x_{i+(M-1)n} + \cdots + x_{i+Mn-1})$$
$$= y_i + y_{i+n} + \cdots + y_{i+(M-2)n} + y_{i+(M-1)n},$$
$$y_{i+(M-1)n} = t - y_i - y_{i+n} - \cdots - y_{i+(M-2)n} \qquad (i = 1, 2, \ldots, n),$$

i.e. $y_{(M-1)n+1}, y_{(M-1)n+2}, \ldots, y_{Mn}$ are linear combinations of $y_1, y_2, \ldots, y_{(M-1)n}$ and t which are independent variables for the first $(M-1)n$ equations of (6) are mutually independent, and the last equation of (6) contains an independent variable y_{Mn} which does not appear in the first $(M-1)n$ equations of (6). Thus (2) can be written in the form

$$(7) \qquad \sum_{j=1}^{n}\left[\sum_{i=0}^{M-2} a_{in+j} F(y_{in+j}, t) + a_{(M-1)n+j} F\left(t - \sum_{i=0}^{M-2} y_{in+j}, t\right)\right] + a' F(0, t) + a'' F(t, t) = 0 .$$

Suppose that a certain $a_{(M-1)n+j_0} \neq 0$ (or $a_{i_0 n+j_0} \neq 0$) exists in (7). By fixing the variables y_{in+j} $(i = 1, 2, \ldots, M—2;\ j = 1, 2, \ldots, j_0-1,\ j_0+1, \ldots, n)$ and denoting the sum of all summands of (7), except $j = j_0$, by $-Ma_{(M-1)n+j_0}\alpha(t)$, then we have

$$\sum_{i=0}^{M-2} a_{in+j_0} F(y_{in+j_0}, t) + a_{(M-1)n+j_0} F\left(t - \sum_{i=0}^{M-2} y_{in+j_0}, t\right) = Ma_{(M-1)n+j_0}\alpha(t) .$$

4° If $a_{in+j_0} = a_{(M-1)n+j_0}$ $(i = 0, 1, 2, \ldots, M-2)$ then the last equation has the general continuous solution

$$F(y, t) = k(t)\left(y - \frac{t}{M}\right) + \alpha(t) .$$

Putting this into (7), we have

$$\sum_{j=1}^{n}\sum_{i=0}^{M-2}(a_{in+j} - a_{(M-1)n+j})\left(y_{in+j} - \frac{t}{M}\right)k(t) + [(M-1)a'' - a']\frac{t}{M}k(t) + \sum_{j=1}^{n}\left[\sum_{i=0}^{M-2} a_{in+j} + a_{(M-1)n+j}\right]\alpha(t) + (a' + a'')\alpha(t) = 0 .$$

Therefore, if $a_{in+j} = a_{(M-1)n+j}$ $(i = 0, 1, \ldots, M-2;\ j = 1, 2, \ldots, n)$ and if

4° (a) $a' = (M-1)a'' \sum_{j=1}^{n} a_{(M-1)n+j} + a'' = 0$, then

$$F(y, t) = k(t)\left(y - \frac{t}{M}\right) + \alpha(t);$$

4° (b) $M\sum_{j=1}^{n} a_{(M-1)n+j} + a' + a'' \neq 0$, then

$$F(y, t) = k(t)\left(y - \frac{t}{M}\right) + \frac{a' - (M-1)a''}{a' + a'' + M\sum_{j=1}^{n} a_{(M-1)n+j}} \cdot \frac{t}{M} k(t) .$$

5° If a certain $a_{i_0 n+j_0}$ exists such that $a_{i_0 n+j_0} \neq a_{(M-1)n+j_0}$, and

$$a' + a'' + \sum_{i=1}^{Mn} a_i = 0 ,$$

then (7) has only the solution $F(y, t) = \alpha(t)$.

By putting above results into $f(x, y) = F(x, x+y)$, we have the

T h e o r e m 2. *The general continuous solution of the functional equation* (1) *for* $n+m=Mn\,(M>2)$ *has the following forms:*

1. $f(x, y)=\left(x-\frac{x+y}{M}\right)k(x+y)+\alpha(x+y)$ *if* $a_{in+j}=a_{(M-1)n+j}$ $(i=0, 1, \ldots, M-2;\ j=1, 2, \ldots, n)$, $a'=(M-1)a''$ *and* $a''+\sum_{j=1}^{n} a_{(M-1)n+j}=0$;

2. $$f(x, y)=\left(x-\frac{x+y}{M}\right)k(x+y)+\frac{a'+(1-M)a''}{a'+a''+M\sum_{j=1}^{n} a_{(M-1)n+j}}\cdot\frac{x+y}{M}\,k(x+y),$$

if $a_{in+j}=a_{(M-1)n+j}$ $(i=0, 1, \ldots, M-2;\ j=1, 2, \ldots, n)$ *and*

$$a'+a''+\sum_{j=1}^{n} a_{(M-1)n+j}\neq 0;$$

3. $f(x, y)=\alpha(x+y)$ *if a certain* $a_{i_0 n+j_0}$ *exists such that* $a_{i_0 n+j_0}\neq a_{(M-1)n+j_0}$, *and* $a'+a''+\sum_{i=1}^{Mn} a_i=0$;

4. $f(x, y)=0$, *in other cases,*

where $k(t)$ *and* $\alpha(t)$ *are arbitrary continuous functions.*

(iii) Now we consider the cases $n+m=Mn+r$ $(1\leqslant r\leqslant n-1)$. Let in (2)

$$x_i+x_{i+1}+\cdots+x_{i+n-1}=y_i \quad (i=1, 2, \ldots, Mn+r), \tag{8}$$
$$x_1+x_2+\cdots+x_{Mn+r}=t.$$

This is a linear system of the algebraic equations. By subtracting certain M equations from the last equation given above, we obtain the equivalent system of the equations

$$x_i+x_{i+1}+\cdots+x_{i+r-1}=t-\sum_{i=1}^{n} y_{i+r+n(j-1)}=Y_i,$$
$$x_1+x_2+\cdots+x_{Mn+r}=t. \tag{9}$$
$$(i=1, 2, \ldots, Mn+r;\quad j=1, 2, \ldots, Mn+r-1).$$

There are two cases

1. $Mn+r=M_1 r$. This is the second case which we have considered above;

2. $Mn+r=M_1 r+r_1$ $(1\leqslant r_1\leqslant r-1)$. In this case we reduce (9) to a simpler equivalent system of the equations by using the method as that of changing (8) to (9).

For the sake of better understanding we give an example.

Let us take the following equation

$$f(x_i+x_{i+1}+\cdots+x_{i+9},\ x_{i+10}+x_{i+11}+x_{i+12}+x_{i+13})=0 \quad (x_{14+j}=x_j). \tag{10}$$

Let $f(x, y)=F(x, x+y)$, we have

$$F(x_i+\cdots+x_{i+9},\ x_1+x_2+\cdots+x_{14})=0. \tag{11}$$

We introduce the new variables

$$\begin{aligned} x_i+x_{i+1}+\cdots+x_{i+9}&=y_i \quad (i=1,\ 2,\ \ldots,\ 14), \\ x_1+x_2+\cdots+x_{14}&=t. \end{aligned} \tag{12}$$

They are equivalent to the system of the equations

$$\begin{aligned} x_i+x_{i+1}+x_{i+2}+x_{i+3}&=t-y_{i+4}=Y_i \quad (i=1,\ 2,\ \ldots,\ 14), \\ x_1+x_2+\cdots+x_{14}&=t. \end{aligned}$$

If we subtract $Y_{i+2}+Y_{i+6}+Y_{i+10}$, from the last equation given above, then we obtain the equivalent system of the equations

$$\begin{aligned} x_i+x_{i+1}&=-2t+y_{i+6}+y_{i+10}+y_i=z_i \quad (i=1,\ 2,\ \ldots,\ 14), \\ x_1+x_2+\cdots+x_{14}&=t. \end{aligned} \tag{13}$$

By (13) we have

$$\begin{aligned} t&=z_1+z_3+z_5+z_7+z_9+z_{11}+z_{13} \\ &=-14t+3y_1+3y_3+3y_5+3y_7+3y_9+3y_{11}+3y_{13}, \\ y_{13}&=5t-y_1-y_3-y_5-y_7-y_9-y_{11} \end{aligned}$$

and

$$\begin{aligned} t&=z_2+z_4+z_6+z_8+z_{10}+z_{12}+z_{14} \\ &=-14t+3y_2+3y_4+3y_6+3y_8+3y_{10}+3y_{12}+3y_{14}, \\ y_{14}&=5t-y_2-y_4-y_6-y_8-y_{10}-y_{12}, \end{aligned}$$

where $y_1,\ y_2,\ \ldots,\ y_{12}$ and t are independent variables. Putting y_{13}, y_{14} and (12) into (10), we have

$$\sum_{i=1}^{6} F(y_{2i-1},\ t)+F\left(5t-\sum_{i=1}^{6} y_{2i-1},\ t\right)=-\left[\sum_{i=1}^{6} F(y_{2i},\ t)+F\left(5t-\sum_{i=1}^{6} y_{2i},\ t\right)\right]=0$$

and its general continuous solution is

$$F(y,\ t)=k(t)\left(y-\frac{5}{7}t\right).$$

If we put this into $f(x, y)=F(x, x+y)$, we obtain the general continuous solutin of (10)

$$f(x,\ y)=k(x+y)\left[x-\frac{5}{7}(x+y)\right]$$

where $k(t)$ is an arbitrary continuous function.

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