PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE MATHÉMATIQUES ET PHYSIQUE

№ 99 (1963)

NOTE ON QUADRATIC FORMS

Svetozar Kurepa

(Received 11 january 1963)

1. Introduction

The object of this note is to derive the well known *Jacobi* formulae ([2] and [1]). These formulae give an explicite way of reducing a quadratic form to its cannonical form. Our derivation is based on the fact that a positive definite self-adjoint operator can be used to introduce a new scalar product. It seems that this method is not widely known and that it throughs a new light on the theory of positive definite quadratic forms and on pairs of quadratic forms. In addition to this in the theory of unitary spaces it enables one to derive the *Jacobi* formulae in few lines.

2. Positive definite quadratic forms

Let $\Phi=\{\alpha, \beta, \ldots\}$ denote the field of reals or the field of all complex numbers. A quadratic form (a hermitian quadratic form) is a function of n-variables $\xi_1, \xi_2, \ldots, \xi_n$ of the form:

$$\varphi=\sum_{i,j=1}^{n} \alpha_{ij}\,\bar{\xi}_i\,\xi_j \tag{1}$$

where $\alpha_{ij}=\bar{\alpha}_{ji}$ are elements of Φ and $\bar{\alpha}$ denotes the complex conjugate of α.

A quadratic form φ is said to be positive definite if $\varphi>0$ for all $\xi_1, \ldots, \xi_n \in \Phi$ and $\varphi=0$ implies $\xi_1=\cdots=\xi_n=0$.

Together with (1) one considers a unitary n-dimensional vector space $X=\{x, y, \ldots\}$ over Φ with a scalar product (x, y). If $e_1, e_2, \ldots, e_n$ is an orthonormal basic set in X then (1) can be written in the form:

$$\varphi=(Ax, x) \tag{2}$$

where $x=\sum_{i=1}^{n} \xi_i\, e_i$ and A is a linear operator defined by

$$A\,e_k=\sum_{i=1}^{n} \alpha_{ik}\, e_i .$$

Since the matrix of A in the basic set $e_1, \ldots, e_n$ is hermitian A is self-adjoint. If the form φ is positive definite then

$$\varphi=(Ax, x)>0 \tag{3}$$

for any $x \in X$, $x \neq 0$, i.e. the self-adjoint operator A is positive definite.

Conversely, if $A > 0$ (i.e. A is a positive definite self-adjoint operator) then (3) in an orthonormal basic set gives a positive definite quadratic form. Hence with a positive definite quadratic form a positive definite self-adjoint operator A is associated in such a way that (3) holds. These are well known facts. Now we set:

$$<x, y> = (Ax, y)$$

for $x, y \in X$. It is obvious that $<x, y>$ is a scalar product in X. In this scalar product, which we call a new scalar product, the quadratic form (3) is written in the form:

$$\varphi = <x, x> = \sum_{k=1}^{n} |<e_k', x>|^2 \tag{4}$$

where $e_1', e_2', \ldots, e_n'$ is any orthonormal basic set in the new scalar product. Already from (4) we see that φ is a sum of squares of linear forms $x \to <x, e_k'>$.

Now, for e_k' we take the orthonormal basic set which is obtained from $e_1, e_2, \ldots, e_n$ by the Gram-Schmidt method of orthogonalisation in the new scalar product.

Hence we have:

$$e_1' = \frac{e_1}{<e_1, e_1>^{1/2}}, \quad e_k' = \frac{\begin{vmatrix} \Gamma(e_1, e_2, \ldots, e_{k-1}) & \begin{matrix} e_1 \\ e_2 \\ \vdots \end{matrix} \\ <e_k, e_1> \cdots <e_k, e_{k-1}> & e_k \end{vmatrix}}{[\Gamma(e_1, \ldots, e_{k-1}) \Gamma(e_1, \ldots, e_k)]^{1/2}} \tag{5}$$

where

$$\Gamma(e_1, \ldots, e_k) = \begin{vmatrix} <e_1, e_1> & <e_1, e_2> & \cdots & <e_1, e_k> \\ <e_2, e_1> & <e_2, e_2> & & <e_2, e_k> \\ \vdots & & & \\ <e_k, e_1> & <e_k, e_2> & & <e_k, e_k> \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{k1} \\ \alpha_{12} & \alpha_{22} & & \alpha_{k2} \\ \vdots & & & \\ \alpha_{1k} & \alpha_{2k} & & \alpha_{kk} \end{vmatrix} \qquad (k = 2, 3, \ldots, n)$$

is the Gram determinant in the new scalar product ([2], p. 239).

We introduce the following notation:

$$\Delta_0 = 1, \quad \Delta_1 = \alpha_{11}, \quad \Delta_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{n1} & & \alpha_{nn} \end{vmatrix}. \tag{6}$$

As we see in the case of positive definite quadratic form any main minor of Δ_n is the Gram determinant of some linearly independent vectors.

Thus any main minor of any order of Δ_n is a positive number. From (5) and (4) we get:

$$\varphi = \sum_{k=1}^{n} \frac{|\eta_k|^2}{\Delta_{k-1}\Delta_k} \tag{7}$$

where

$$\eta_1 = A_1(x), \ldots, \eta_k = \begin{vmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{k-1,1} & A_1(x) \\ \alpha_{12} & \alpha_{22} & & \alpha_{k-1,2} & A_2(x) \\ \vdots & & & & \\ \alpha_{1k} & \alpha_{2k} & & \alpha_{k-1,k} & A_k(x) \end{vmatrix} \tag{8}$$

and

$$A_p(x) = (e_p, Ax) = \sum_{j=1}^{n} \overline{\xi}_j (Ae_p, e_j), \qquad \text{i.e.}$$

$$A_p(x) = \sum_{j=1}^{n} \alpha_{jp} \overline{\xi}_j \qquad p = 1, 2, \ldots, n. \tag{9}$$

By (7), (8) and (9) the explicite formulae for reducing φ to its cannonical form are given*.

3. Quadratic forms of the rank r

The formulae (7), (8) and (9) have the meaning also if φ is not necessarily positive definite provided that $\Delta_1, \Delta_2, \ldots, \Delta_n$ do not vanish.

Suppose that (1) is quadratic form of the rank r, i.e. the matrix of coefficients (α_{ij}) has the rank r and that

$$\Delta_1 = \alpha_{11} \neq 0, \quad \Delta_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \neq 0, \ldots, \Delta_r = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & & \\ \alpha_{r1} & & \alpha_{rr} \end{vmatrix} \neq 0. \tag{10}$$

We assert that in this case

$$\varphi = \sum_{k=1}^{r} \frac{|\eta_k|^2}{\Delta_{k-1}\Delta_k}$$

where η_k $(k = 1, 2, \ldots, r)$ are given by (8).

In order to prove this we observe that

$$A(\lambda) = A + \lambda I$$

is positive definite for $\lambda > \lambda_0$, where I is the identity operator and λ_0 a suitable real number.

Thus:

$$(Ax, x) + \lambda(x, x) = \sum_{k=1}^{n} \frac{|\eta_k(\lambda)|^2}{\Delta_{k-1}(\lambda)\Delta_k(\lambda)} \tag{11}$$

* The formulae (7), (8) and (9) were rediscovered and proved differently by *D. Blanuša* [1] whose lectures to students in 1962 initiated this investigations.

where

$$(12) \qquad \Delta_k(\lambda) = \begin{vmatrix} \alpha_{11}+\lambda & \alpha_{21} & \cdots & \alpha_{k1} \\ \alpha_{12} & \alpha_{22}+\lambda & & \alpha_{k2} \\ \vdots & & & \\ \alpha_{1k} & \alpha_{2k} & & \alpha_{kk}+\lambda \end{vmatrix}$$

and

$$(13) \qquad \eta_k(\lambda) = \begin{vmatrix} \alpha_{11}+\lambda & \alpha_{21} & \cdots & \alpha_{k-1,1} & A_1(x)+\lambda\overline{\xi_1} \\ \alpha_{12} & \alpha_{22}+\lambda & & \alpha_{k-1,2} & A_2(x)+\lambda\overline{\xi_2} \\ \vdots & & & & \\ \alpha_{1k} & \alpha_{2k} & & \alpha_{k-1,k} & A_k(x)+\lambda\overline{\xi_k} \end{vmatrix}$$

$$= \sum_{j=1}^{n} D_{kj}(\lambda)\overline{\xi_j}$$

with

$$(14) \qquad D_{kj}(\lambda) = \begin{vmatrix} \alpha_{11}+\lambda & \alpha_{21} & \cdots & \alpha_{k-1,1} & \alpha_{j1}+\lambda\delta_{j1} \\ \alpha_{12} & \alpha_{22}+\lambda & & \alpha_{k-1,2} & \alpha_{j2}+\lambda\delta_{j2} \\ \vdots & & & & \\ \alpha_{1k} & \alpha_{2k} & & \alpha_{k-1,k} & \alpha_{jk}+\lambda\delta_{jk} \end{vmatrix}.$$

Now, the right side of (11) is the ratio of two polinomials in λ and (11) holds for all $\lambda > \lambda_0$. If we make analytic continuation, i.e. if we take λ complex ($|\eta_k(\lambda)|^2$ is assumed to be written as the polinomial in $\lambda > \lambda_0$) then (11) holds for all complex λ. We take $\lambda \downarrow 0$ and we get:

$$(15) \qquad (Ax, x) = \sum_{k=1}^{r} \frac{|\eta_k|^2}{\Delta_{k-1}\Delta_k} + \lim_{\lambda\downarrow 0} \sum_{k=r+1}^{n} \frac{|\eta_k(\lambda)|^2}{\Delta_{k-1}(\lambda)\Delta_k(\lambda)}.$$

It remains to prove that:

$$(16) \qquad \lim_{\lambda\downarrow 0} \frac{|\eta_k(\lambda)|^2}{\Delta_{k-1}(\lambda)\Delta_k(\lambda)} = 0 \quad \text{for} \quad r < k < n.$$

To prove this we observe that the coefficient of λ^p in the polinomial $\Delta_k(\lambda)$ is proportional to the sum of all main minors of $\Delta_k = \Delta_k(0)$ of the order $k-p$. Since Δ_k has the rank r we see that

$$(17) \qquad \Delta_k(\lambda) = a_k\lambda^{k-r} + b_k\lambda^{k-r+1} + \cdots$$

where a_k, as the sum of all main minors of Δ_k of the order r, does not vanish*.

Furthermore from (14) we see that $D_{kj}(\lambda) = 0$ if $j < k$, $D_{kk}(\lambda) = \Delta_k(\lambda)$ and for $j > k$ we have:

$$D_{kj}(\lambda) = D_{kj}(0) + \frac{D'_{kj}(0)}{1!} + \lambda^2\frac{D''_{kj}(0)}{2!} + \cdots$$

$$= \lambda^{k-r}\frac{D^{(k-r)}(0)}{(k-r)!} + \cdots;$$

* Observe that the dimension of the null-subspace of a self-adjoint operator H is equal to the multiplicity of zero as the root of $\det(\lambda I - H) = 0$.

$D_{kj}(0)$ is the sum of minors of the matrix (α_{ij}) of the order k. Since $k > r$ it vanishes. In the same way $D'_{kj}(0)$ as the sum of minors of the order $k-1$ vanishes if $k-1 > r$ etc. Hence:

$$\eta_k(\lambda) = c_k \lambda^{k-r} + d_k \lambda^{k-r+1} + \cdots \tag{18}$$

where c_k may vanish. Now (17) and (18) imply:

$$\frac{|\eta_k(\lambda)|^2}{\Delta_{k-1}(\lambda)\,\Delta_k(\lambda)} = \lambda\,\frac{|c_k|^2 + \lambda(c_k \bar{d}_k + \bar{c}_k d_k) + \cdots}{a_{k-1} a_k + \lambda(a_{k-1} b_k + a_k b_{k-1}) + \cdots} \qquad (\lambda > 0)$$

from which (16) follows and therefore:

$$\varphi = \sum_{k=1}^{r} \frac{|\eta_k|^2}{\Delta_{k-1}\Delta_k}. \tag{19}$$

Since the forms $\eta_1, \ldots, \eta_r$ are linearly independent the *Jacobi* formulae (19), (10), (9) and (8) give the explicit reduction of the form φ to its cannonical form. If $r = n$ then (19) implies that φ is positive definite if and only if $\Delta_{k-1}\Delta_k > 0$, i.e. if all minors (6) are positive. Thus the fact that minors $\Delta_1, \Delta_2, \ldots, \Delta_n$ are positive implies that φ is positive definite and therefore that all main minors of (α_{ij}) are positive. Furthermore for $r = n$ (19) implies that $-\varphi$ is positive definite, i.e. φ negative definite, if and only if $\Delta_{k-1}\Delta_k < 0$, i.e. if $(-1)^k \Delta_k > 0$ $(k = 1, 2, \ldots, n)$.

4. Pairs of quadratic forms

In this section we prove the well known theorem that two quadratic forms:

$$\varphi = \sum_{i,j=1}^{n} \alpha_{ij} \bar{\xi}_i \xi_j, \qquad \psi = \sum_{i,j=1}^{n} \beta_{ij} \bar{\xi}_i \xi_j \qquad (\alpha_{ij} = \bar{\alpha}_{ji},\ \beta_{ij} = \bar{\beta}_{ji}), \tag{20}$$

can be brought with a same linear transformation to the cannonical form provided that φ is positive definite. We write (20) in the form:

$$\varphi = (Ax, x) \qquad \psi = (Bx, x) \qquad x \in X \tag{21}$$

where A and B are self-adjoint operators in an n-dimensional unitary space X. Furthermore A is positive definite.

If we set $<x, y> = (Ax, y)$ then (21) becomes:

$$\varphi = <x, x>, \qquad \psi = <Dx, x> \qquad (D = A^{-1}B). \tag{22}$$

If $C: X \to X$ is any linear operator, C^* the adjoint of C in the scalar product $(\ )$ and C^+ the adjoint of C in the scalar product $<\ >$, then

$$<x, C^+ y> = <Cx, y> = (ACx, y) = (x, C^* Ay)$$

and

$$<x, C^+ y> = (Ax, C^+ y) = (x, AC^+ y)$$

imply

$$AC^+ = C^* A, \quad \text{i.e.}$$

$$C^+ = A^{-1} C^* A. \tag{23}$$

According to (23) we have:

$$A^{+} = A$$

and

$$D^{+} = B^{+} A^{-1} = (A^{-1} BA) A^{-1} = A^{-1} B = D, \quad \text{i.e.}$$

the operator D is self-adjoint in the new product.

Since D is self-adjoint there is an orthonormal basic set $e_1'', \ldots, e_n''$ such that

$$D e_k'' = d_k e_k''$$

where d_k are real numbers. In this basic set (22) becomes:

$$\varphi = \sum_{k=1}^{n} |\langle e_k'', x\rangle|^2 \quad \text{and} \quad \varphi = \sum_{k=1}^{n} d_k |\langle e_k'', x\rangle|^2 \tag{24}$$

which proves that φ and ψ, by the same linear (not necessarily unitary) transformation can be brought to the sum of squares of linear forms. The real numbers d_k are roots of the equation $\det(\lambda I - D) = 0$, i.e. of the equation $\det(\lambda I - A^{-1} B) = 0$. Thus $d_1, \ldots, d_k$ are roots of the equation

$$\det(\lambda A - B) = 0.$$

REFERENCES

[1] D. Blanuša:
Obrat formule za ortogonaliziranje, Rad Hrv. AZU, knjiga 276 (86) (1945), 62—74.

[2] F. R. Gantmacher:
Matrizenrechnung I, Berlin 1958.