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A SPECIAL FUNCTIONAL EQUATION

L. Carlitz

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Mitrinović and *Prešić* **[1]** have proved that the general solution of the functional equation

$$f(x, y)\, f(u, v) + f(x, u)\, f(v, y) + f(x, v)\, f(y, u) = 0 \tag{1}$$

is given by

$$f(x, y) = g(x)\, h(y) - g(y)\, h(x), \tag{2}$$

where $g(x)$, $h(x)$ are arbitrary functions of x.

In the present note we consider the functional equation

$$\begin{aligned} f(x, y, z)\, f(u, v, w) &+ f(y, x, u)\, f(z, v, w) \\ &+ f(x, y, v)\, f(z, u, w) + f(y, x, w)\, f(z, u, v) = 0. \end{aligned} \tag{3}$$

The variables and the functional values are assumed to be complex numbers.

If we take $x = y = z = u = v = w$, it is clear that (3) implies

$$f(x, x, x) = 0. \tag{4}$$

Next if we take $x = y = z = u$, $v = w$, we get

$$\begin{aligned} f(x, x, x)\, f(x, v, v) &+ f(x, x, x)\, f(x, v, v) \\ &+ f(x, x, v)\, f(x, x, v) + f(x, x, v)\, f(x, x, v) = 0, \end{aligned}$$

so that

$$f(x, x, v) = 0. \tag{5}$$

If we take $x = u = v = w$, $y = z$, we get

$$2 f^2(y, x, x) + f(x, y, x)\, f(y, x, x) = 0, \tag{6}$$

while $x = z$, $y = u = v = w$ yields

$$f^2(x, y, y) + 2 f(x, y, y)\, f(y, x, y) = 0.$$

Interchanging x and y, this becomes

$$f^2(y, x, x) + 2 f(x, y, x)\, f(y, x, x) = 0; \tag{7}$$

comparison of (7) with (6) gives

$$f(y, x, x) = 0. \tag{8}$$

In the next place, if we take $x=u,\ y=z=v=w$, we get

$$f(x,y,y)f(x,y,y)+f(y,x,x)f(y,y,y)$$
$$+f(x,y,y)f(y,x,y)+f(y,x,y)f(y,x,y)=0.$$

Making use of (8), this becomes

(9) $$f(y,x,y)=0.$$

Now let $a,\ b,\ c$ be fixed complex numbers such that

(10) $$f(a,b,c)\neq 0.$$

If we take $v=w$ in (3) we get

$$(f(x,y,v)+f(y,x,v))f(z,u,v)=0.$$

In particular, for $z=a,\ u=b,\ v=c$, this implies

(11) $$f(x,y,c)=-f(y,x,c).$$

If we take $z=w$ in (3) we get

$$f(x,y,z)f(u,v,z)+f(y,x,z)f(z,u,v)=0.$$

For $x,y,z=a,b,c$ this becomes, in view of (10) and (11),

(12) $$f(u,v,c)=f(c,u,v).$$

For $z=v$ we find that

$$f(x,y,z)f(u,z,w)+f(x,y,z)f(z,u,w)=0,$$

which implies

(13) $$f(c,u,w)=-f(u,c,w).$$

It is evident from the above, that

$$f(a',b',c')\neq 0,$$

where $a',\ b',\ c'$ is any permutation of $a,\ b,\ c$. Thus, in particular (11), (12), (13) hold when c is replaced by a or b. It follows, for example, that

$$f(a,b,u)=-f(b,a,u).$$

We now define

(14) $$\Phi_1(u)=\frac{f(a,b,u)}{f(a,b,c)},$$

(15) $$\Psi(u,v)=f(u,v,c).$$

Then it follows from (3), (11), (12) and (13) that

(16) $$f(u,v,w)=\Phi_1(u)\Psi(v,w)-\Phi_1(v)\Psi(u,w)+\Phi_1(w)\Psi(u,v).$$

Also, if we take $x=u=c$ in (3), we get

(17) $$\Psi(y,z)\Psi(v,w)+\Psi(y,v)\Psi(w,z)+\Psi(y,w)\Psi(z,v)=0.$$

Comparing (17) with (1) we infer that

$$\Psi(u, v) = \Phi_2(u)\,\Phi_3(v) - \Phi_2(v)\,\Phi_3(u),$$

where $\Phi_2(u)$, $\Phi_3(u)$ are arbitrary functions. Therefore (16) becomes

$$f(u, v, w) = \begin{vmatrix} \Phi_1(u) & \Phi_1(v) & \Phi_1(w) \\ \Phi_2(u) & \Phi_2(v) & \Phi_2(w) \\ \Phi_3(u) & \Phi_3(v) & \Phi_3(w) \end{vmatrix}. \tag{18}$$

Conversely, if $f(u, v, w)$ is defined by means of (18), then (3) is satisfied. This follows on expanding the vanishing determinant

$$\begin{vmatrix} \Phi_1(x) & \Phi_2(x) & \Phi_3(x) & 0 & 0 & 0 \\ \Phi_1(y) & \Phi_2(y) & \Phi_3(y) & 0 & 0 & 0 \\ \Phi_1(z) & \Phi_2(z) & \Phi_3(z) & \Phi_1(z) & \Phi_2(z) & \Phi_3(z) \\ \Phi_1(u) & \Phi_2(u) & \Phi_3(u) & \Phi_1(u) & \Phi_2(u) & \Phi_3(u) \\ \Phi_1(v) & \Phi_2(v) & \Phi_3(v) & \Phi_1(v) & \Phi_2(v) & \Phi_3(v) \\ \Phi_1(w) & \Phi_2(w) & \Phi_3(w) & \Phi_1(w) & \Phi_2(w) & \Phi_3(w) \end{vmatrix}.$$

We have therefore proved the following.

Theorem. — *The general complex solution of the functional equation* (3) *is given by* (18), *where* $\Phi_1(u)$, $\Phi_2(u)$, $\Phi_3(u)$ *are arbitrary complex functions.*

REFERENCE

[1] D. S. Mitrinović and S. B. Prešić:
Sur une équation fonctionnelle cyclique d'ordre supérieur, ces Publications № 70, 1962.