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ON THE GENERALISATION OF THE SARRUS' RULE

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We shall show here how Sarrus' rule can be generalized and applied for the evaluation of the determinants of order n.

Sarrus' rule

Usually in almost every textbook for the determinants, the practical Sarrus' rule for the evaluation of determinants of order 3 can be found.

This rule consists in the following

Scheme (1)

$$\begin{array}{|ccc|cc} \multicolumn{1}{c}{+} & + & \multicolumn{1}{c}{+} & & \\ \multicolumn{1}{c}{\searrow} & \searrow & \multicolumn{1}{c}{\searrow} & & \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ \multicolumn{1}{c}{\nearrow} & \nearrow & \multicolumn{1}{c}{\nearrow} & & \\ \multicolumn{1}{c}{-} & - & \multicolumn{1}{c}{-} & & \end{array}$$

To rewrite the first two columns to the right.

To perform the product of elements in the direction of arrows.

To take the signs plus and minus alternatively as shown by scheme (1).

The determinant is then equal to the sum of 6 terms so obtained.

Remark: Neither the rewriting to the right is essential (the second and third columns can be rewritten to the left), nor the rewriting of columns (the first and second rows can be added under the scheme, or the second and third above the scheme).

Rebić's analogue to the Sarrus' rule

There exists a procedure analogous to the Sarrus' rule[1] although it is not its generalisation, formulated by D. Rebić in this way (cf. scheme (2)).

Scheme (2)

$$
\begin{array}{cccc|ccc}
+ & - & + & - & & & \\
\searrow & \searrow & \searrow & \searrow & & & \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43} \\
\nearrow & \nearrow & \nearrow & \nearrow & & & \\
+ & - & + & - & & &
\end{array}
\qquad
\begin{array}{cccc|ccc}
+ & - & + & - & & & \\
\searrow & \searrow & \searrow & \searrow & & & \\
a_{11} & a_{13} & a_{14} & a_{12} & a_{11} & a_{13} & a_{14} \\
a_{21} & a_{23} & a_{24} & a_{22} & a_{21} & a_{23} & a_{24} \\
a_{31} & a_{33} & a_{34} & a_{32} & a_{31} & a_{33} & a_{34} \\
a_{41} & a_{43} & a_{44} & a_{42} & a_{41} & a_{43} & a_{44} \\
\nearrow & \nearrow & \nearrow & \nearrow & & & \\
+ & - & + & - & & &
\end{array}
$$

$$
\begin{array}{cccc|ccc}
+ & - & + & - & & & \\
\searrow & \searrow & \searrow & \searrow & & & \\
a_{11} & a_{14} & a_{12} & a_{13} & a_{11} & a_{14} & a_{12} \\
a_{21} & a_{24} & a_{22} & a_{23} & a_{21} & a_{24} & a_{22} \\
a_{31} & a_{34} & a_{32} & a_{33} & a_{31} & a_{34} & a_{32} \\
a_{41} & a_{44} & a_{42} & a_{43} & a_{41} & a_{44} & a_{42} \\
\nearrow & \nearrow & \nearrow & \nearrow & & & \\
+ & - & + & - & & &
\end{array}
$$

1. To perform the cyclic permutation of the second, third and fourth columns, the first column remaining at the same place in the determinant.

2. To rewrite the first three columns to the right for all three determinants, which are obtained by the cyclic permutation and to perform the product of elements in the direction of arrows.

3. To take the signs plus and minus alternatively as shown in scheme (2).

4. The sum of all 24 terms so obtained gives the value of the determinant.

We draw attention to the fact that Rebić's procedure can't be generalized even to the fifth-order determinants.

Namely, if we should arrange a scheme for the determinants of order five, analogous to the scheme (2), permutating cyclicly, the second, third, fourth and fifth columns, the scheme would then contain four determinants. From each of them, according to the Rebić's procedure, one gets ten products, i. e. altogether 40 terms of the determinant the value of which is to be found.

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[1] This procedure has been published in the book:
D. S. Mitrinović: Zbornik Matematičkih Problema I (second revised and completed edition, Beograd 1958), page 258, problem 78.

But, according to the definition of the determinant, it must have $5! = 120$ terms.

Nevertheless, Rebić's procedure has suggested to us the following generalisation of the Sarrus' rule.

Generalized Sarrus' rule

The formulation of the general procedure for the evaluation of the determinant D of the order n $(n \geqslant 3)$ is:

$$D = |a_{ij}| \qquad (i, j = 1, 2, 3, \ldots, n). \tag{3}$$

1. To write down $\frac{1}{2}(n-1)!$ square schemes (analogous to these ones from (2) or (4)) which are to be obtained from (3) permutating the columns from the second to the $n-th$ (under the only condition not to take two permutations whose indices proceed in inverse order), the first column remaining at the same place of the determinant (cf. the schemes (2) and (4)).

2. To rewrite the first $n-1$ columns to the right to all of $\frac{1}{2}(n-1)!$ determinants obtained with the shown permutations of columns and carry out the multiplication of elements along the traced arrows (as in (1), (2) and (4)).

3. To discern two cases depending on the fact whether the determinant D is of even or odd order:

Determinant D is of even order

With the product of elements along the arrows directed down:

If the scheme is obtained (according to 2.) by an even number of permutations of columns, the signs plus and minus advance alternatively (cf. (2)).

If the number of permutations is odd, the signs minus and plus go alternatively.

Along the arrows directed upward:

If the number $n/2$ is even, the signs are the same as with the corresponding arrrows pointing down. If $n/2$ is odd, the signs are opposite to those of the corresponding arrows down.

Determinant D is of odd order

The sign along the arrows down is:

plus if the number of permutations of columns is even; minus if it is odd (examples (1) and (4)).

The sign of the product along the arrows pointing upward is:

minus if the number of permutation is even and $\frac{1}{2}(n-1)$ odd, or inversely the number of permutations is odd and $\frac{1}{2}(n-1)$ even;

plus if both the number of permutations and the number $\frac{1}{2}(n-1)$ are even or both odd.

4. The determinant D is equal to the sum of $n!$ terms so obtained.

We draw the attention that so formulated generalized Sarrus' rule contains as particular cases both Sarrus' rule and Rebić's procedure.

Scheme (4)

$$\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{41} & a_{42} & a_{43} & a_{44} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{51} & a_{52} & a_{53} & a_{54} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
+ & + & + & + & + & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{13} & a_{15} & a_{14} & a_{11} & a_{12} & a_{13} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{25} & a_{24} & a_{21} & a_{22} & a_{23} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{35} & a_{34} & a_{31} & a_{32} & a_{33} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{45} & a_{44} & a_{41} & a_{42} & a_{43} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{55} & a_{54} & a_{51} & a_{52} & a_{53} & a_{55} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}$$

$$\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{14} & a_{13} & a_{15} & a_{11} & a_{12} & a_{14} & a_{13} \\
a_{21} & a_{22} & a_{24} & a_{23} & a_{25} & a_{21} & a_{22} & a_{24} & a_{23} \\
a_{31} & a_{32} & a_{34} & a_{33} & a_{35} & a_{31} & a_{32} & a_{34} & a_{33} \\
a_{41} & a_{42} & a_{44} & a_{43} & a_{45} & a_{41} & a_{42} & a_{44} & a_{43} \\
a_{51} & a_{52} & a_{54} & a_{53} & a_{55} & a_{51} & a_{52} & a_{54} & a_{53} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{14} & a_{15} & a_{13} & a_{11} & a_{12} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{24} & a_{25} & a_{23} & a_{21} & a_{22} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{34} & a_{35} & a_{33} & a_{31} & a_{32} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{44} & a_{45} & a_{43} & a_{41} & a_{42} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{54} & a_{55} & a_{53} & a_{51} & a_{52} & a_{54} & a_{55} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
+ & + & + & + & + & & & &
\end{array}$$

$$\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{15} & a_{13} & a_{14} & a_{11} & a_{12} & a_{15} & a_{13} \\
a_{21} & a_{22} & a_{25} & a_{23} & a_{24} & a_{21} & a_{22} & a_{25} & a_{23} \\
a_{31} & a_{32} & a_{35} & a_{33} & a_{34} & a_{31} & a_{32} & a_{35} & a_{33} \\
a_{41} & a_{42} & a_{45} & a_{43} & a_{44} & a_{41} & a_{42} & a_{45} & a_{43} \\
a_{51} & a_{52} & a_{55} & a_{53} & a_{54} & a_{51} & a_{52} & a_{55} & a_{53} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
+ & + & + & + & + & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{12} & a_{15} & a_{14} & a_{13} & a_{11} & a_{12} & a_{15} & a_{14} \\
a_{21} & a_{22} & a_{25} & a_{24} & a_{23} & a_{21} & a_{22} & a_{25} & a_{24} \\
a_{31} & a_{32} & a_{35} & a_{34} & a_{33} & a_{31} & a_{32} & a_{35} & a_{34} \\
a_{41} & a_{42} & a_{45} & a_{44} & a_{43} & a_{41} & a_{42} & a_{45} & a_{44} \\
a_{51} & a_{52} & a_{55} & a_{54} & a_{53} & a_{51} & a_{52} & a_{55} & a_{54} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}$$

$$\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{13} & a_{12} & a_{14} & a_{15} & a_{11} & a_{13} & a_{12} & a_{14} \\
a_{21} & a_{23} & a_{22} & a_{24} & a_{25} & a_{21} & a_{23} & a_{22} & a_{24} \\
a_{31} & a_{33} & a_{32} & a_{34} & a_{35} & a_{31} & a_{33} & a_{32} & a_{34} \\
a_{41} & a_{43} & a_{42} & a_{44} & a_{45} & a_{41} & a_{43} & a_{42} & a_{44} \\
a_{51} & a_{53} & a_{52} & a_{54} & a_{55} & a_{51} & a_{53} & a_{52} & a_{54} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{13} & a_{12} & a_{15} & a_{14} & a_{11} & a_{13} & a_{12} & a_{15} \\
a_{21} & a_{23} & a_{22} & a_{25} & a_{24} & a_{21} & a_{23} & a_{22} & a_{25} \\
a_{31} & a_{33} & a_{32} & a_{35} & a_{34} & a_{31} & a_{33} & a_{32} & a_{35} \\
a_{41} & a_{43} & a_{42} & a_{45} & a_{44} & a_{41} & a_{43} & a_{42} & a_{45} \\
a_{51} & a_{53} & a_{52} & a_{55} & a_{54} & a_{51} & a_{53} & a_{52} & a_{55} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
+ & + & + & + & + & & & &
\end{array}$$

$$\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{13} & a_{14} & a_{12} & a_{15} & a_{11} & a_{13} & a_{14} & a_{12} \\
a_{21} & a_{23} & a_{24} & a_{22} & a_{25} & a_{21} & a_{23} & a_{24} & a_{22} \\
a_{31} & a_{33} & a_{34} & a_{32} & a_{35} & a_{31} & a_{33} & a_{34} & a_{32} \\
a_{41} & a_{43} & a_{44} & a_{42} & a_{45} & a_{41} & a_{43} & a_{44} & a_{42} \\
a_{51} & a_{53} & a_{54} & a_{52} & a_{55} & a_{51} & a_{53} & a_{54} & a_{52} \\
+ & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
\nwarrow & + & + & + & + & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{13} & a_{14} & a_{15} & a_{12} & a_{11} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{23} & a_{24} & a_{25} & a_{22} & a_{21} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{33} & a_{34} & a_{35} & a_{32} & a_{31} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{43} & a_{44} & a_{45} & a_{42} & a_{41} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{53} & a_{54} & a_{55} & a_{52} & a_{51} & a_{53} & a_{54} & a_{55} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}$$

$$\begin{array}{ccccc|cccc}
- & - & - & - & - & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{13} & a_{15} & a_{12} & a_{14} & a_{11} & a_{13} & a_{15} & a_{12} \\
a_{21} & a_{23} & a_{25} & a_{22} & a_{24} & a_{21} & a_{23} & a_{25} & a_{22} \\
a_{31} & a_{33} & a_{35} & a_{32} & a_{34} & a_{31} & a_{33} & a_{35} & a_{32} \\
a_{41} & a_{43} & a_{45} & a_{42} & a_{44} & a_{41} & a_{43} & a_{45} & a_{42} \\
a_{51} & a_{53} & a_{55} & a_{52} & a_{54} & a_{51} & a_{53} & a_{55} & a_{52} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
- & - & - & - & - & & & &
\end{array}
\qquad
\begin{array}{ccccc|cccc}
+ & + & + & + & + & & & & \\
\searrow & \searrow & \searrow & \searrow & \searrow & & & & \\
a_{11} & a_{14} & a_{12} & a_{13} & a_{15} & a_{11} & a_{14} & a_{12} & a_{13} \\
a_{21} & a_{24} & a_{22} & a_{23} & a_{25} & a_{21} & a_{24} & a_{22} & a_{23} \\
a_{31} & a_{34} & a_{32} & a_{33} & a_{35} & a_{31} & a_{34} & a_{32} & a_{33} \\
a_{41} & a_{44} & a_{42} & a_{43} & a_{45} & a_{41} & a_{44} & a_{42} & a_{43} \\
a_{51} & a_{54} & a_{52} & a_{53} & a_{55} & a_{51} & a_{54} & a_{52} & a_{53} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & & \\
+ & + & + & + & + & & & &
\end{array}$$

This scheme (4) on the pages 20 and 21 has been derived from the generalized Sarrus' rule (for the particular case $n = 5$) by taking first 12 permutations of columns.

Proof: We recall that according to the definition, the value of the determinant D is to be found in this way:

To take the product of elements along the main diagonal of the determinant D

$$a_{11}\, a_{22}\, a_{33} \, \ldots \, a_{nn}$$

and retaining the first indices unchanged, to permutate the second indices in all possible manners.

Each of the $n!$ products so obtained is to be taken with the sign plus, if the number of the permutations of second indices is of even class in it, and with the sign minus if the number of these permutations is odd.

Summing up all such products one gets the value of the determinant D.

We have to prove now that the products of the elements of the determinant so formed, and the products which had been derived from our general procedure are to be identical.

In order to show it, notice that in each of $\frac{1}{2}(n-1)!$ schemes formed according to the general Sarrus' rule there exist two products of this sort: their first indices (showing the number of rows) proceed in natural order, and their second indices (showing the number of columns) begin with the number $k\,(1 \leqslant k \leqslant n)$.

One of these two products is placed along the arrow directed down. It must occur in each of the mentioned schemes, for according to paragraph 2. of the general Sarrus' rule, all schemes contain the same second indices (permutated only) including the index k.

The second product is placed along the arrow pointing upward and occurs because the first $n-1$ columns are to be rewritten to the right together with the index k (if the number k is in the last column of the square scheme of the determinant, one of the arrows finishes in it and the second starts from it).

Hence we conclude that because there are two products whose second indices begin with k in each of $\frac{1}{2}(n-1)!$ schemes, their total number will be $2 \cdot \frac{1}{2}(n-1)! = (n-1)!$.

And owing to the fact that $1 \leqslant k \leqslant n$, i. e. there are n indices k, the sum of all products amounts to $n \cdot (n-1)! = n!$, which is in concordance with the number of terms obtained from the definition of determinant given above.

The second indices of these products (the second indices of which begin with k) are the same as the indices of columns which the arrows (on which these products lie) traverse; that is why these indices (from the second on) of the arrow pointing down, proceed in inverse order than of the arrow going upward.

That is the reason why we have included in paragraph 1. the condition for the permutations; without this restriction we could get equal products proceeding as shown in paragraph 1.

It still remains to prove that the paragraph 3. of the general Sarrus' rule, about the signs, is correct.

That results from the following facts:

When the first (resp. last) index of the second indices along whichever arrow pointing down (upward) is displaced at the last (first) place (by means of $n-1$ inverses), one gets the order of the second indices of the next arrow pointing down (upward).

The second indices along the first arrow directed upward proceed in the inverse order than the indices of the first arrow pointing down.

The number $n/2$ (if D is of even order) or $(n-1)/2$ (if D is of odd order) has to be added to the number of inversions in the product along the arrow directed down to get the number of inversions with the product of the corresponding arrow pointing upward.

Then, for instance suppose the determinant D is of even order, the number of permutations of columns in the considered scheme is odd, and the number $n/2$ is odd. It results that the sign of the product along the first arrow directed down is minus, the signs of the products along the following parallel arrows proceed alternatively. The sign of the product along the first arrow pointing upward is plus, the following signs go alternatively.

In connection with the Sarrus' rule for determinants of order 3, there have appeared various articles. Recently R. Osborn[2] has tried to prove that Sarrus' rule can't be generalized to the determinants of higher order.

That is why we considered it of interest to show that however, developping already mentioned idea of D. Rebić, Sarrus' rule can be generalized and applied to the evaluation of the determinants of order n.

Added in proofs: V. Zani[3], according to the reference of Kochendörfer in „Jahrbuch über die Fortschritte der Mathematik", Band 63_{II}, Jahrgang 1937, Heft 1, Seite 850, has performed the generalisation of Sarrus' rule to the fourth-order determinants and pointed out the generalisation to the determinants of order n. Unfortunately, we couldn't get Zani's paper till now.

Rezime

O UOPŠTENJU SARRUS-OVOG PRAVILA

Slobodan V. Pavlović

U ovom radu pokazano je kako se Sarrus-ovo pravilo može generalisati i primeniti na izračunavanje determinanata reda n.

Generalizacija je izvedena na osnovu ideje D. Rebića da pri izračunavanju vrednosti determinanata četvrtog reda upotrebi ne jednu shemu kao kod Sarrus-ovog pravila već tri analogne sheme (videti (2)).

No pri toj generalizaciji ne mogu se formirati sheme kao kod Rebićevog postupka, cikličnom permutacijom kolona; moraju se ispisati $\frac{1}{2}(n-1)!$ kvadratnih shema (analognih sa shemama (2) ili (4)) koje se dobijaju iz (3) permutacijom kolona od druge do n-te (pod jedinim uslovom da se ne uzmu dve permutacije čiji indeksi teku u obrnutom redu), dok prva kolona ostaje na istom mestu u determinanti.

Pitanje znakova proizvoda analognih sa proizvodima iz Sarrus-ove ili Rebićeve sheme se tada lakše rešava.

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[2] *R. Osborn*: Concerning fourth-order Determinants, American Mathematical Monthly, vol. 67, № 7. p. 682—683 (1960).

[2] *V. Zani:* Generalizzazione della regola de Sarrus, Bollettino matematico, Firenze, (2) 16, 1—5.