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PARTIAL MATCHINGS AND PATTERN AVOIDANCE

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A partition of a finite set all of whose blocks have size one or two is called a *partial matching*. Here, we enumerate classes of partial matchings characterized by the avoidance of a single pattern, specializing a natural notion of partition containment that has been introduced by Sagan. Let $v_n(\tau)$ denote the number of partial matchings of size n which avoid the pattern τ. Among our results, we show that the generating function for the numbers $v_n(\tau)$ is always rational for a certain infinite family of patterns τ. We also provide some general explicit formulas for $v_n(\tau)$ in terms of $v_n(\rho)$, where ρ is a pattern contained in τ. Finally, we find, with two exceptions, explicit formulas and/or generating functions for the number of partial matchings avoiding any pattern of length at most five.

1. INTRODUCTION

A *partition* of a finite set is any collection of non-empty, mutually disjoint subsets, called *blocks*, whose union is the set. (There is a single empty partition of the empty set which has no blocks.) From now on, we will use the term partition when referring to a partition of a finite set. We will denote the set of all partitions of $[n] = \{1, 2, \ldots, n\}$ by P_n and the set of all partitions of $[n]$ containing exactly k blocks by $P_{n,k}$ (note $[0] = \varnothing$). A *partial matching* of $[n]$, also called an *involution*, is any member of P_n all of whose blocks contain either one or two elements. The set of all partial matchings of $[n]$ will be denoted by V_n and the set of all partial matchings of $[n]$ containing exactly k blocks by $V_{n,k}$; note that $V_{n,k} = \varnothing$ if $k < n/2$. Let $v_n = |V_n|$ and $v_{n,k} = |V_{n,k}|$ for $n, k \geq 1$, with $v_0 = v_{0,0} = 1$. Then $v_n = \sum_{k=0}^{n} v_{n,k}$, where the $v_{n,k}$, called *Bessel numbers*, are given by the explicit formula

$$v_{n,k} = \frac{n!}{2^{n-k}(2k-n)!(n-k)!}, \qquad n/2 \leq k \leq n,$$

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see, e.g., [**5**]. The n-th Bessel polynomial $y_n(x)$ may be expressed using Bessel numbers as $y_n(x) = \sum_{k=0}^{n} v_{n+k,n} x^k$ and satisfies the differential equation

$$x^2 y'' + (2x+2)y' = n(n+1)y,$$

see, e.g., [**2, 8, 14**]. For further information on the sequences v_n and $v_{n,k}$, see, respectively, [**20**, A000085] and [**20**, A144299]. In this paper, we will be enumerating various restricted subsets of V_n and $V_{n,k}$ characterized by the avoidance of certain patterns.

A partition Π is said to be in *standard form* if it is written as $\Pi = B_1/B_2/\cdots$, where $\min(B_1) < \min(B_2) < \cdots$. One may also represent the partition $\Pi = B_1/\cdots/B_k \in P_{n,k}$, equivalently, by the *canonical sequential form* $\pi = \pi_1 \cdots \pi_n$, wherein $j \in B_{\pi_j}$, $1 \le j \le n$ (see, e.g., [**21**]). For example, the partition $\Pi = 1,2,7/3,5,10/4,8/6,9 \in P_{10,4}$ has the canonical sequential form $\pi = 1123241342$, and in such case we write $\Pi = \pi$. Note that $\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}$ is a *restricted growth function* from $[n]$ to $[k]$ (see, e.g., [**17**] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \le \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \le i \le n-1$.

In what follows, we will represent partial matchings by their canonical sequential forms as described above for partitions. Note that a sequential form $\pi = \pi_1\pi_2\cdots$ of a partition corresponds to a partial matching if and only if each letter appears either once or twice. For example, $\Pi = 1,4/2/3,6/5 \in V_{6,4}$ is given by $\pi = 123143$. We now define the notion of avoidance. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_m$ be two partial matchings, represented by their canonical sequences. We say that σ *contains* τ if σ contains a subsequence that is order-isomorphic to τ; that is, σ has a subsequence $\sigma_{f(1)}, \sigma_{f(2)}, \ldots, \sigma_{f(m)}$, where $1 \le f(1) < f(2) < \cdots < f(m) \le n$, such that for each $i, j \in [m]$, we have $\sigma_{f(i)} < \sigma_{f(j)}$ if and only if $\tau_i < \tau_j$ and $\sigma_{f(i)} > \sigma_{f(j)}$ if and only if $\tau_i > \tau_j$. Otherwise, we say that σ *avoids* τ. In this context, τ is usually called a *pattern*. For example, the partial matching σ avoids the pattern 1221 if there exist no indices $i < j < k < \ell$ with $\sigma_i = \sigma_\ell < \sigma_j = \sigma_k$ and avoids 1232 if there exist no such indices with $\sigma_i < \sigma_j = \sigma_\ell < \sigma_k$.

The concept of pattern-avoidance described above is the restriction to partial matchings of Sagan's concept of avoidance on finite set partitions [**18**]. See also Klazar [**12**], Chen et al [**3**], Goyt [**7**], and Jelínek and Mansour [**9**] for further work on the pattern avoidance question for partitions. The avoidance problem on involutions obtained by restricting the usual avoidance problem on permutations (represented as words), considered first by Knuth [**13**] and Simion and Schmidt [**19**], is studied in [**6**] (see also the references therein), where the Wilf equivalence classes are determined for patterns up to length seven. In [**10**], the authors considered the avoidance problem on *complete matchings*, i.e., partitions all of whose blocks have size two, and Wilf equivalence classes were deduced for patterns up to length seven. See also the related paper by Chen et al [**4**].

We will use the following notation. If τ is a pattern, then let $V_n(\tau)$ and $V_{n,k}(\tau)$ denote the subsets of V_n and $V_{n,k}$, respectively, all of whose members avoid τ. We will denote the cardinalities of $V_n(\tau)$ and $V_{n,k}(\tau)$ by $v_n(\tau)$ and $v_{n,k}(\tau)$, respectively. From the definitions, note that $v_n(\tau) = \sum_{k\geq 0} v_{n,k}(\tau)$. We will say that two patterns τ and σ are (Wilf) *equivalent*, denoted by $\tau \sim \sigma$, if $v_n(\tau) = v_n(\sigma)$ for all $n \geq 0$.

In the next section, we prove some general enumerative results concerning pattern avoidance by partial matchings, represented canonically. Among our results is the equivalence of the patterns $12\cdots ki$ and $12\cdots(k-1)ik$ for all i and k and the fact that the generating function $\sum_{n\geq 0} v_n(12\cdots ki)x^n$ is always rational. We also present several ways of deriving an enumerative formula for $v_n(\tau)$ from one for $v_n(\rho)$, where ρ is some shorter pattern contained in τ. In the subsequent two sections, we then supply an explicit formula for the number $v_n(\tau)$ and/or its generating function in the cases when τ has length four or five. We use both algebraic and combinatorial techniques to establish our results. In the cases of 1123 and 12331, we use the kernel method to solve the functional equations that arise once certain parameters related to the patterns in question have been introduced. In the former case, an explicit formula results which may then be explained combinatorially, while in the latter case, no such formula seems to exist, though it is possible to write an expression for the generating function showing it is algebraic.

We will employ the following notation: if $\tau = \tau_1, \tau_2, \ldots, \tau_m$ is a sequence of numbers and i is an integer, then $\tau + i$ refers to the sequence $\tau_1 + i, \tau_2 + i, \ldots, \tau_m + i$. Also, if m and n are positive integers, then $[m, n]$ denotes the set $\{m, m+1, \ldots, n\}$ if $m \leq n$, with $[m, n] = \varnothing$ if $m > n$. Throughout, let F_n denote the sequence of Fibonacci numbers defined by $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$. Henceforth, the terms *involution* and *partial matching* will be used interchangeably.

2. SOME GENERAL RESULTS

Before we present the main results, we first state, without proof, three preliminary observations.

Observation 1. *The set $V_n(12\cdots k)$ is empty for $n \geq 2k-1$.*

Observation 2. *A partial matching π of length n avoids the pattern 121 if and only if π has the form $1^{\alpha_1}2^{\alpha_2}\cdots$, where $\alpha_1 + \alpha_2 + \cdots = n$ and $\alpha_i \in \{1, 2\}$ for all i. Therefore, $v_n(121) = F_{n+1}$ if $n \geq 0$. Furthermore, we have $v_{n,r}(121) = \binom{r}{n-r}$ for all n and r, where $r \in [n]$.*

Observation 3. *Similarly, we have $v_n(112) = F_{n+1}$ and $v_{n,r}(112) = \binom{r}{n-r}$, the members of $V_{n,r}(112)$ being of the form $\pi = 12\cdots rr^{\beta_r}(r-1)^{\beta_{r-1}}\cdots 1^{\beta_1}$ and $\beta_1 + \beta_2 + \cdots + \beta_r = n - r$, with $\beta_i \in \{0, 1\}$ for all i.*

The next result provides a way showing how from a given pair of equivalent patterns we can construct new equivalent pairs of longer patterns.

Proposition 4. *Let σ be a non-empty pattern. If $n \geq 2$, then*

$$v_n(1(\sigma+1)) = v_{n-1}(\sigma) + (n-1)v_{n-2}(\sigma). \tag{1}$$

Thus, if $f(x) = \sum_{n\geq 0} v_n(\sigma)x^n$, then the generating function $g(x) = \sum_{n\geq 0} v_n\big(1(\sigma+1)\big)x^n$ is given by

$$g(x) = 1 + (x+x^2)f(x) + x^3 f'(x). \tag{2}$$

In particular, if α and β are two patterns and $\alpha \sim \beta$, then $1(\alpha+1) \sim 1(\beta+1)$.

Proof. Since each involution has either one occurrence of the letter 1 or two occurrences, we obtain (1). Note that in the second case, there are $n-1$ possible positions for the second 1. Formula (2) follows from multiplying (1) by x^n, summing over $n \geq 2$, and noting initial values. The last statement follows from either of the first two. □

Example 5. Note that $v_n(11) = 1$ for all $n \geq 0$. Using (1) repeatedly, we obtain

$$\begin{aligned}
v_n(122) &= v_{n-1}(11) + (n-1)v_{n-2}(11) = 1 + n - 1 = n, \quad n \geq 1,\\
v_n(1233) &= v_{n-1}(122) + (n-1)v_{n-2}(122) = n - 1 + (n-1)(n-2) = (n-1)^2, \ n \geq 3,\\
v_n(12344) &= (n-2)^2 + (n-1)(n-3)^2 = (n-1)(n-2)(n-3) + 1, \quad n \geq 5,\\
v_n(123455) &= (n-2)(n-3)(n-4) + 1 + (n-1)\big((n-3)(n-4)(n-5) + 1\big)\\
&= n^4 - 12n^3 + 50n^2 - 80n + 36, \quad n \geq 7.
\end{aligned}$$

By induction, one sees that $v_n(12\cdots kk)$ is a polynomial in n of degree $k-1$ and that the coefficient of n^{k-1} in $v_n(12\cdots kk)$ is 1.

Example 5 together with the observations above are sufficient to enumerate the involutions avoiding a pattern of length three.

$\tau\backslash n$	4	5	6	7	8	9	Reference
112, 121	5	8	13	21	34	55	Observations 2 and 3
122	4	5	6	7	8	9	Example 5
123	3	0	0	0	0	0	Observation 1

Table 1. Values of $v_n(\tau)$, where τ is a pattern of length three.

Our next result is a general equivalence.

Theorem 6. *If $k \geq 2$ and $i \in [k-1]$, then $12\cdots ki \sim 12\cdots(k-1)ik$.*

Proof. If $2 \leq i \leq k-1$, then $12\cdots(k-i+1)1 \sim 12\cdots(k-i)1(k-i+1)$ implies $12\cdots ki \sim 12\cdots(k-1)ik$, by Proposition 4 (applied $i-1$ times). Thus, it is enough to consider the case when $i=1$. Let $\sigma = 12\cdots k1$ and $\tau = 12\cdots(k-1)1k$. We will define an explicit bijection f between $V_{n,m}(\sigma)$ and $V_{n,m}(\tau)$ for all n and m.

If $1 \leq m \leq k-1$, then we may take f to be the identity, so assume $m \geq k$. Let us write $\pi = 1w_1 2w_2\cdots mw_m \in V_{n,m}(\sigma)$, where each w_i is a word in $[i]$. We will call an occurrence of τ in an involution ρ in which the letter 1 in τ corresponds to the actual letter j in ρ a *j-occurrence* of τ, and, similarly, for σ. Suppose π contains a j-occurrence of τ; note that $1 \le j \le m-(k-1)$ since π contains m blocks. Then the second j of π must occur in the word w_{j+k-2}, for if the second j were in w_ℓ for some $\ell < j+k-2$, then there would not exist a j-occurrence of τ in π, and if it were in w_ℓ for some $\ell > j+k-2$, then there would be a j-occurrence of σ in π and there isn't.

Now suppose that π has j-occurrences of τ for $j = j_1, j_2, \ldots, j_r$, where $j_1 < j_2 < \cdots < j_r$. Then j_i lies in the word w_{j_i+k-2} for each $i \in [r]$, that is, $w_{j_i+k-2} = \alpha_i j_i \beta_i$, where α_i and β_i are (possibly empty) words in $[j_i+k-2]$. Observe further that α_i and β_i are actually words in $[j_i+1, j_i+k-2]$ since π avoids σ. Let $S = \{j_1+k-2, j_2+k-2, \ldots, j_r+k-2\}$; note that $m \notin S$. Let π' be the involution obtained from π as follows:

(i) If $\ell \in [m-1]-S$, then leave the word w_ℓ unchanged in π,

(ii) If $\ell = j_i+k-2$ for some $i \in [r]$, then replace the word $w_{j_i+k-2} = \alpha_i j_i \beta_i$ with α_i,

(iii) Replace the word w_m with the (concatenated) word $w_m j_r \beta_r j_{r-1} \beta_{r-1} \cdots j_1 \beta_1$.

One may verify that the mapping $\pi \mapsto \pi'$ is a bijection from $V_{n,m}(\sigma)$ to $V_{n,m}(\tau)$. Note that the mapping is reversed upon considering the word w'_m in $\pi' = 1w'_1 2w'_2 \cdots m w'_m$, where each w'_i is i-ary. First decompose w'_m as $w'_m = \gamma y_1 \gamma_1 y_2 \gamma_2 \cdots y_r \gamma_r$, where γ consists only of letters greater than or equal to $m-(k-2)$, $y_1 \le m-(k-1)$, and each y_i, $i > 1$, represents the i-th left-to-right minimum in the subword of w'_m consisting of all letters past y_1, inclusive. To reverse the mapping $\pi \mapsto \pi'$, insert the word $y_i \gamma_i$ just before the first occurrence of the letter $y_i + k - 1$ for each $i \in [r]$ and replace the word w'_m with γ. $\square$

We now show that the generating function for the numbers $v_n(12\cdots k1) = v_n(12\cdots(k-1)1k)$ is always rational.

Theorem 7. *Let $k \ge 2$. Then the generating function $f_k(x) = \sum_{n\ge 0} v_n(12\cdots k1)x^n$ is a rational function.*

Proof. If $\tau = \tau_1\tau_2\cdots\tau_i$ is a sequence, then we will denote the largest term of τ by $m(\tau)$ and the number of terms in τ by $|\tau|$, that is, $m(\tau) = \max_{1\le j\le i}\{\tau_j\}$ and $|\tau| = i$. Let T_k be the set of all non-empty involutions τ with $m(\tau) \le k-1$, that is, $T_k = \cup_{i=1}^{2k-2}\left(\cup_{j=1}^{k-1} V_{i,j}\right)$. We denote the generating function for the number of involutions $\pi = \pi_1\pi_2\cdots\pi_n$ of length n that avoid $12\cdots k1$ with $\pi_1\pi_2\cdots\pi_i = \tau_1\tau_2\cdots\tau_i$ by $f_{k,\tau_1\tau_2\cdots\tau_i}(x)$. We let $f_{k,\emptyset}(x) = f_k(x)$. From the definitions, we have

$$f_k(x) = 1 + f_{k,1}(x), \quad f_{k,\tau}(x) = x^{|\tau|} + \sum_{j=1}^{m(\tau)+1} f_{k,\tau j}(x), \quad \tau \in T_k.$$

Note that $f_{k,\tau j}(x) = 0$ when j already appears twice in τ, since no letter can occur three times in a matching. Also, if the letter 1 appears twice in τ, then $f_{k,\tau}(x) = x^2 f_{k,\tau'}(x)$, where τ' is the involution obtained from τ by deleting the two occurrences of the letter 1 and decreasing any other letters by 1. We now consider τ of the following three forms:

(i) $\tau = 1(\tau'+1) \in T_k$,

(ii) $\tau = 1(\tau'+1)1 \in T_k$,

(iii) $\tau = 1(\tau'+1)k$, where $1(\tau'+1) \in V_{i,k-1}$ for some i.

In case (iii), note that $f_{k,\tau}(x) = x f_{k,\tau'(k-1)}(x)$, since involutions starting with τ and avoiding $12\cdots k1$ cannot have a second 1.

We now write a linear system of equations in the variables $f_k(x)$ and $f_{k,\tau}(x)$, where τ has one of the three forms above. Our equations are $f_k(x) = 1 + f_{k,1}(x)$ and

$$f_{k,\tau}(x) = \begin{cases} x^{|\tau|} + \sum_{j=1}^{m(\tau)+1} f_{k,\tau j}(x), & \text{if } \tau \text{ is of form (i);} \\ x^2 f_{k,\tau'}(x), & \text{if } \tau \text{ is of form (ii);} \\ x f_{k,\tau'(k-1)}(x), & \text{if } \tau \text{ is of form (iii).} \end{cases}$$

Note that in case (iii), if $\tau'(k-1)$ contains two 1's and does not end in 1, then the pattern $\tau'(k-1)$ may be reduced until it is empty or is of the form (i) or (ii) above. This implies that the generating functions $f_k(x)$ and $f_{k,\tau}(x)$, where τ is of one of the three forms above, satisfy a system of linear equations having a solution whose coefficients are polynomials in x.

Let $C = C(x)$ denote the coefficient matrix corresponding to the aforementioned system of equations. Then the determinant $|C(x)|$ is non-zero for all x sufficiently close to zero. To see this, suppose that the equations for $f_{k,\tau}(x)$ are written from top to bottom in ascending order according to the length of τ. If $x = 0$, then C is upper triangular with all 1's on the main diagonal, which implies $|C(0)| = 1$. By continuity, there exists an interval containing the origin over which $|C(x)|$ is non-zero, as desired.

For such x, the system above then has a unique solution. By Cramer's Theorem, this solution is a vector of rational functions in x. In particular, $f_k(x) = \sum_{n\geq 0} v_n(12\cdots k1)x^n$ is a rational function. $\square$

Combining formula (2) with the prior theorem yields the following result.

Corollary 8. *The generating functions* $\sum_{n\geq 0} v_n(12\cdots ki)x^n$ *are rational for all* $1 \leq i \leq k$.

Example 9. If $k = 2$, then the proof of the prior theorem gives

$$f_2(x) = 1 + f_{2,1}(x), \quad f_{2,1}(x) = x + f_{2,11}(x) + f_{2,12}(x),$$

$$f_{2,11}(x) = x^2 f_2(x), \quad f_{2,12}(x) = x f_{2,1}(x).$$

Solving this system implies $f_2(x) = \dfrac{1}{1-x-x^2}$.

When $k=3$, we have

$$f_3(x) = 1 + f_{3,1}(x), \quad f_{3,1}(x) = x + f_{3,11}(x) + f_{3,12}(x), \quad f_{3,11}(x) = x^2 f_3(x),$$
$$f_{3,12}(x) = x^2 + f_{3,121}(x) + f_{3,122}(x) + f_{3,123}(x), \quad f_{3,121}(x) = x^2 f_{3,1}(x),$$
$$f_{3,122}(x) = x^3 + f_{3,1221}(x) + f_{3,1223}(x), \quad f_{3,123}(x) = x f_{3,12}(x),$$
$$f_{3,1221}(x) = x^2 f_{3,11}(x), \quad f_{3,1223}(x) = x f_{3,112}(x) = x^3 f_{3,1}(x).$$

Solving this system implies $f_3(x) = \dfrac{1-x^2}{1-x-2x^2-x^4}$.

Similarly, we have $f_4(x) = \dfrac{1-2x^2-x^3-2x^4-x^6}{1-x-3x^2-2x^4-x^5-5x^6-x^8}$.

The following results provide ways of deriving enumerative formulas for longer patterns from shorter ones, thereby extending equivalences to pairs of patterns of greater length.

Theorem 10. *Suppose σ and τ are patterns both having two occurrences of the symbol* 1, *with $\sigma \sim \tau$. Then $11(\sigma+1) \sim 11(\tau+1)$. Furthermore, we have*

$$v_n(11(\sigma+1)) = 1 + \sum_{r=1}^{n-1}\sum_{i=1}^{r} r\binom{r-1}{i-1} v_{n-r-i}(\sigma) \prod_{j=1}^{i-1}(n-r-j),$$

where $v_m(\sigma) = 0$ if $m < 0$.

Proof. We prove the second statement. Let $\sigma' = 11(\sigma+1)$. First note that $\pi = 12\cdots n \in V_n(\sigma')$. Otherwise, let us count the members $\pi = \pi_1\pi_2\cdots\pi_n \in V_n(\sigma')$ of the form $\pi = 12\cdots rj\pi'$, where $r \in [n-1]$ is fixed and $j \in [r]$. Let us condition further on the number i of letters in $[r]$ that are repeated in π. Note first that there are r choices for the element j and $\binom{r-1}{i-1}$ choices for the additional letters in $[r]$ that are to be repeated.

Now observe that a member of $[r]$ within the involution of the given form can act only as a 1 within a possible occurrence of σ', as there are two occurrences of the symbol 2 in $\sigma+1$ and there are no two occurrences of a letter a preceding the two occurrences of a letter b, where $1 \le a < b \le r$. Thus, there are

$$(n-r-1)(n-r-2)\cdots(n-r-(i-1))$$

choices regarding the positions within π for the additional $i-1$ letters in $[r]$ that are to be repeated. Finally, the remaining $n-r-i$ positions of π are to be filled with letters in $\{r+1, r+2, \ldots\}$, and the involution comprising these positions must avoid σ due to the two occurrences of the letter $j \le r$. That is, there are $v_{n-r-i}(\sigma)$ choices for these positions. Conversely, any involution of the form $\pi = 12\cdots rj\pi'$

and having exactly i repeated members of $[r]$, where the other letters are arranged as described above, is seen to avoid σ'. Thus, there are $r\binom{r-1}{i-1}v_{n-r-i}(\sigma)\prod_{j=1}^{i-1}(n-r-j)$ such involutions, and summing over all possible r and i gives the result.

Theorem 11. *Suppose σ and τ are patterns both having two occurrences of the symbol 1, with $\sigma \sim \tau$. Then $121(\sigma+2) \sim 121(\tau+2)$. Furthermore, we have*

$$\begin{aligned} &v_n(121(\sigma+2)) \\ &\quad = F_{n+1} + \sum_{r=1}^{n-2}\sum_{j=r}^{2r}\sum_{i=1}^{2r-j}(2r-j)\binom{r}{j-r}\binom{2r-j}{i-1}v_{n-j-i-1}(\sigma)\prod_{t=2}^{i}(n-j-t) \\ &\quad + \sum_{r=1}^{n-3}\sum_{j=r}^{2r}\sum_{i=1}^{2r-j}(2r-j)\binom{r}{j-r}\binom{2r-j-1}{i-1}v_{n-j-i-2}(\sigma)\prod_{t=3}^{i+1}(n-j-t), \end{aligned}$$

where $v_m(\sigma) = 0$ if $m < 0$.

Proof. We prove the second statement. Let $\sigma' = 121(\sigma+2)$. Note first that $V_n(121) \subseteq V_n(\sigma')$, with $v_n(121) = F_{n+1}$, by Observation 2. So suppose $\pi = \pi_1\pi_2\cdots\pi_n \in V_n(\sigma') - V_n(121)$ is of the form

$$\pi = 1^{\alpha_1}2^{\alpha_2}\cdots(r+1)^{\alpha_{r+1}}\ell\pi',$$

where $r \in [n-2]$ is fixed, $\alpha_s = 1$ or 2 for $s \in [r+1]$, and $\ell \in [r]$.

First suppose $\alpha_{r+1} = 1$. Let $j = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and i be the number of members of $[r+1]$ that occur in $\ell\pi'$. Note first that there are $\binom{r}{j-r}$ choices for the indices $s \in [r]$ for which $\alpha_s = 2$. There are then $2r-j$ choices for the letter ℓ in π and $\binom{2r-j}{i-1}$ choices for the letters in $[r+1]$ that occur in π'. There are then $\prod_{t=2}^{i}(n-j-t)$ ways to arrange these letters within the $n-j-2$ positions of π' and $v_{n-j-i-1}(\sigma)$ choices for the remaining letters of π', which belong to $\{r+2, r+3, \ldots\}$ and comprise a member of $V_{n-j-i-1}(\sigma)$ (on those letters). Summing over all possible r, j and i gives the first sum on the right-hand side above. Similar reasoning applies to the case when $\alpha_{r+1} = 2$ and gives the second sum, which completes the proof. $\square$

We will say that the two involution patterns σ and τ are *strongly equivalent* (following the terminology used in [**10**] in conjunction with complete matchings), if there exists a bijection f between the sets of σ-avoiding and τ-avoiding involutions with the property that for any σ-avoiding involution λ, the number of blocks of λ is equal to the number of blocks of $f(\lambda)$, and moreover for any i, the i-th block of λ has the same size as the i-th block of $f(\lambda)$. The following result, which we state without proof, may be obtained by modifying slightly the proof of the comparable result for complete matchings found in [**10**, Lemma 3.10].

Proposition 12. *Let σ and τ be strongly equivalent patterns both containing k distinct letters. Let ρ be a pattern that has two occurrences of the symbol* 1. *Then the patterns $\sigma(\rho+k)$ and $\tau(\rho+k)$ are strongly equivalent.*

Example 13. Since the patterns 112 and 121 are seen to be strongly equivalent, so are the patterns $112(\sigma+2)$ and $121(\sigma+2)$, and thus $v_n(112(\sigma+2))$ is given explicitly by the formula in Theorem 11 above, where σ is a pattern having two occurrences of the symbol 1.

Let us say that two involution patterns σ and τ are *block equivalent*, which we'll denote by $\sigma \dot{\sim} \tau$, if there exists a bijection from $V_{n,m}(\sigma)$ to $V_{n,m}(\tau)$ for all n and m. For example, the proof of Proposition 4 above shows further that if $\sigma \dot{\sim} \tau$, then $1(\sigma{+}1) \dot{\sim} 1(\tau{+}1)$, which may be extended to $12\cdots k(\sigma{+}k) \dot{\sim} 12\cdots k(\tau{+}k)$ for any $k \geq 1$. The following result provides another way of extending block equivalence to pairs of patterns of greater length.

Theorem 14. *Suppose σ and τ are non-empty patterns on $[k-1]$, with $\sigma \dot{\sim} \tau$. Then $\sigma kk \dot{\sim} \tau kk$. Furthermore, we have*

$$v_{n,r}(\sigma kk)$$
$$= \sum_{m=0}^{r-1} \sum_{\ell=0}^{n-r-1} (r-m+\ell)\ell! \binom{r-m-1+\ell}{\ell} \binom{m+r+1-n+\ell}{\ell} v_{m+n-r-\ell-1,m}(\sigma),$$

where $1 \leq r \leq n-1$.

Proof. We prove the second statement. Let $\sigma' = \sigma kk$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in V_{n,r}(\sigma')$, where $1 \leq r \leq n-1$, and suppose that $m+1$ is the largest letter which occurs twice in π, where $0 \leq m \leq r-1$. Then we may write π in the form

$$\pi = \pi'(m+1)w, \tag{3}$$

where $\pi' \in V_{\ell,m}(\sigma)$ for some $\ell \geq m$ and w is a word having all distinct letters and containing $m+1$. Conversely, note the any involution of this form indeed avoids the pattern σ'.

To enumerate the σ'-avoiding involutions π of length n having the form (3), first note that there are $v_{\ell,m}(\sigma)$ choices for π' and $n-\ell-1$ choices for the position of the second $m+1$. There are then $\binom{n-\ell-2}{r-m-1}$ ways to choose the positions in w to be occupied by letters in $[m+2,r]$, each of which occurs singly. Once these positions in w are taken, we fill in the remaining positions of w, and for this, we must select $n-\ell-2-(r-m-1) = m+n-r-\ell-1$ letters from those in $[m]$ that were not repeated in π'. There are $m-(\ell-m) = 2m-\ell$ such letters and thus $\binom{2m-\ell}{m+n-r-\ell-1}$ ways to select them and $(m+n-r-\ell-1)!$ ways to arrange them in the remaining positions of w, once selected. Thus, the number of members of $V_{n,r}(\sigma')$ of the form (3) is

$$(n-\ell-1)(m+n-r-\ell-1)! \binom{n-\ell-2}{r-m-1} \binom{2m-\ell}{m+n-r-\ell-1} v_{\ell,m}(\sigma).$$

Summing this over $0 \leq m \leq r-1$ and $m \leq \ell \leq m+n-r-1$, and then replacing ℓ with $m+n-r-1-\ell$, gives the result.

3. AVOIDING A PATTERN OF LENGTH FOUR

In this section, we consider the problem of avoiding a single pattern of length four by involutions. The cases of avoiding either 1212 or 1221 have been previously encountered, see [**16**].

Fact 15. *If $n \geq 1$, then $v_n(1212) = v_n(1221) = m_n$, where m_n denotes the n-th Motzkin number* (*see* [**20**, A001006]).

The cases 1234 or 1233 were covered by Observation 1 and Example 5, respectively. We now consider the cases of avoiding 1223 or 1232.

Proposition 16. *If $n \geq 1$, then $v_n(1223) = v_n(1232) = F_n + (n-1)F_{n-1}$.*

Proof. By (1), we have $v_n(1223) = v_{n-1}(112) + (n-1)v_{n-2}(112)$ if $n \geq 2$. By Observation 3, we have $v_n(112) = F_{n+1}$, which completes the first case. A similar argument applies to the second. $\square$

In the proofs of the next three propositions, we let $a_n = v_n(\tau)$, where τ is the pattern in question, and let $a_{n,j}$ denote the number of members of $V_n(\tau)$ such that the block $\{1, j\}$ occurs if $j > 1$ and the number of members of $V_n(\tau)$ such that the block $\{1\}$ occurs if $j = 1$. Using a different approach, we first consider the cases of avoiding 1231 and 1213, which were done already in Example 9.

Proposition 17. *The generating function for the numbers $v_n(1231)$ or $v_n(1213)$ is given by*

$$\frac{1-x^2}{1-x-2x^2-x^4}.$$

Proof. We first treat the case 1231. From the definitions, we have $a_{n,1} = a_{n-1}$, $a_{n,2} = a_{n,3} = a_{n-2}$, $a_{n,4} = a_{n-4}$, and $a_{n,j} = 0$ for all $5 \leq j \leq n$. Thus, the sequence a_n satisfies

$$a_n = a_{n-1} + 2a_{n-2} + a_{n-4}, \quad n \geq 4,$$

with $a_0 = a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. The rest follows easily. The second case follows from the first and Theorem 6, or can be done directly by similar reasoning. $\square$

We now consider the case of avoiding 1122. Here and elsewhere, we take $\binom{n}{k} = 0$ if $k > n \geq 0$ or if $k < 0$.

Proposition 18. *If $n \geq 1$, then*

$$v_n(1122) = n + \sum_{m=1}^{n-1} \sum_{j=n-1-2m}^{n-1-m} j(n-1-m-j)! \binom{n-1-m}{j} \binom{m}{n-1-m-j}.$$

Proof. Let us assume $n \geq 4$, for the formula is easily seen to hold otherwise. From the definitions, we have $a_{n,1} = a_{n-1}$ and $a_{n,2} = 1$. If the element 2 shares a block with a member of $[3, n]$, then let j' denote this other member, with $j' = 2$ if the block $\{2\}$ occurs. Fix $3 \leq j \leq n$. Within members of $V_n(1122)$ enumerated by $a_{n,j}$, we have $j' \in [2, n] - \{j\}$. Considering whether $j' = 2$ or $3 \leq j' \leq j-1$ or $j+1 \leq j' \leq n$ yields the recurrence

$$a_{n,j} = a_{n-1,j-1} + \sum_{i=3}^{j-1} a_{n-2,i-1} + (n-j)a_{n-2,j-1}, \qquad 3 \leq j \leq n,$$

where $a_{n,2} = 1$ (and so $a_{n,3} = n-2$). Thus, for all $j = 4, 5, \ldots, n$, we have

$$b_{n,j} := a_{n,j} - a_{n,j-1} = b_{n-1,j-1} + (n-j)b_{n-2,j-1}.$$

Define $b'_{n,j} = b_{n,j}/(n-j)!$, so

$$b'_{n,j} = b'_{n-1,j-1} + b'_{n-2,j-1},$$

with $b'_{n,3} = 1/(n-4)!$.

Define $b'_n(u) = \sum_{j=3}^{n} b'_{n,j}u^j$. Therefore,

$$b'_n(u) = \frac{u^3}{(n-4)!} + ub'_{n-1}(u) + ub'_{n-2}(u),$$

for all $n \geq 4$.

Define $b'(v, u) = \sum_{n \geq 4} b'_n(u)v^n$. Therefore,

$$b(v, u) = \frac{u^3v^4e^v}{1 - uv(1+v)} = u^3v^4 \sum_{j\geq 0}\sum_{i \geq 0} \frac{u^i v^{i+j}(1+v)^i}{j!}.$$

Hence,

$$b'_{n,m} = \sum_{j=n+2-2m}^{n-m-1} \frac{1}{j!}\binom{m-3}{2m-2+j-n},$$

which implies

$$b_{n,m} = \sum_{j=n+2-2m}^{n-m-1} (j+1)(n-1-m-j)!\binom{n-m}{j+1}\binom{m-3}{n-m-1-j}.$$

By definition of $b_{n,m}$, we have

$$a_{n,s} = a_{n,3} + \sum_{m=4}^{s} b_{n,m},$$

which implies

$$a_{n,s} = n - 2 + \sum_{m=4}^{s} \sum_{j=n+3-2m}^{n-m} j(n-m-j)! \binom{n-m}{j} \binom{m-3}{n-m-j}.$$

Since $a_{n,n} = a_{n-2}$, we obtain

$$a_n = n + \sum_{m=4}^{n+2} \sum_{j=n+5-2m}^{n+2-m} j(n+2-m-j)! \binom{n+2-m}{j} \binom{m-3}{n+2-m-j},$$

which completes the proof.

Corollary 19. *If $n \geq 1$, then*

$$\sum_{k=0}^{n/2} \binom{n}{2k} k! = n + \sum_{m=1}^{n-1} \sum_{j=n-1-2m}^{n-1-m} j(n-1-m-j)! \binom{n-1-m}{j} \binom{m}{n-1-m-j}.$$

Proof. To show this, we argue combinatorially that the left-hand side counts the members of $V_n(1122)$. Let $Z = Z_k$ denote the subset of $V_n(1122)$ whose members have exactly k doubleton blocks, where $0 \leq k \leq n/2$. We will show that $|Z| = \binom{n}{2k} k!$, whence the result follows from summing over k. To do so, we first argue that the number of members of Z in which 1 belongs to a doubleton block is $\binom{n-1}{2k-1} k!$. Let $S = \{a_1 = 1 < a_2 < \cdots < a_{2k}\}$ denote the set of elements in $[n]$ comprising the doubleton blocks within such a member of Z. Note that no block of the form $\{a_i, a_j\}$ can occur, where $1 \leq i < j \leq k$. Otherwise, each a_r, where $r \in [k+1, 2k]$, would have to be paired with some a_s, where $s \in [k]$, in order to avoid an occurrence of 1122, which is impossible. Therefore, only blocks of the form $\{a_i, a_j\}$, where $1 \leq i \leq k < j \leq 2k$, are possible, which implies that there are $k!$ ways in which to pair the elements of S. Thus, there are $\binom{n-1}{2k-1} k!$ members of Z in which the element 1 belongs to a doubleton block. Similarly, there are $\binom{n-1}{2k} k!$ members of Z in which the block $\{1\}$ occurs and thus $\binom{n-1}{2k-1} k! + \binom{n-1}{2k} k! = \binom{n}{2k} k!$ members of Z altogether, as desired.

Proposition 20. *The generating function for the numbers $v_n(1123)$ is given by*

$$\frac{4(x^3 - x^2 - x + 1)}{\left(x - 1 - \sqrt{1 - 2x - 3x^2 + 4x^3}\,\right)^2}.$$

Proof. Clearly, $a_{n,2} = 0$ if $n \geq 5$, with $a_{n,1} = a_{n-1}$ and $a_{n,n-1} = a_{n,n} = a_{n-2}$. If the element 2 shares a block with a member of $[3, n]$, then let i' denote this other member, with $i' = 2$ if the block $\{2\}$ occurs. Fix $3 \leq i \leq n$. Note that within members of $V_n(1123)$ enumerated by $a_{n,i}$, we have $i' \in [2, i-1] \cup \{n\}$. Considering whether $i' = 2$ or $3 \le i' \le i-1$ or $i' = n$ yields

$$a_{n,i} = a_{n-1,i-1} + \sum_{j=3}^{i-1} a_{n-2,j-1} + a_{n-2,i-1}, \qquad 3 \le i \le n.$$

Let $a_n(u) = \sum_{j=1}^{n} a_{n,j} u^j$; note that $a_n(1) = a_n$. Multiplying the above recurrence by u^i and summing over $i = 3, 4, \ldots, n$ yields

$$a_n(u) - a_{n-1}(1)u = u(a_{n-1}(u) - a_{n-2}(1)u) + \sum_{j=3}^{n-1} \frac{u^j - u^{n+1}}{1-u} a_{n-2,j-1},$$

which implies for $n \ge 5$,

$$\begin{aligned} a_n(u) = a_{n-1}(1)u + ua_{n-1}(u) - a_{n-2}(1)u^2 + \frac{u}{1-u}(a_{n-2}(u) - a_{n-3}(1)u) \\ - \frac{u^{n+1}}{1-u}(a_{n-2}(1) - a_{n-3}(1)). \end{aligned}$$

Let $a(x,u) = \sum_{n\ge 2} a_n(u)x^n$. Multiplying the last recurrence by x^n/u^n, summing over $n \ge 5$, and noting the initial conditions $a_2(u) = u + u^2$, $a_3(u) = 2u + u^2 + u^3$ and $a_4(u) = 4u + u^2 + 2u^3 + 2u^4$, we obtain

$$\begin{aligned} &\frac{u(1-x)(1-u) - x^2}{u(1-u)}(a(x/u,u) - xa(x/u,1)) \\ &\qquad = \frac{(1-x)(1+x)(1+u)x^2}{u} - \frac{ux^2(1-x)}{1-u} a(x,1). \end{aligned}$$

We use the *kernel method* (see, e.g., [**1**]) and substitute $u = \frac{1-x+\sqrt{1-2x-3x^2+4x^3}}{2(1-x)}$ in the above functional equation to cancel out the left-hand side and obtain

$$\begin{aligned} \sum_{n\ge 0} a_n x^n &= 1 + x + a(x,1) = 1 + x + \frac{(1+x)(1-u^2)}{u^2} = \frac{1+x}{u^2} \\ &= \frac{4(x^3 - x^2 - x + 1)}{(x - 1 - \sqrt{1-2x-3x^2+4x^3})^2}, \end{aligned}$$

which completes the proof. □

Using the previous result, one may determine an explicit formula for $v_n(1123)$. Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th Catalan number.

Corollary 21. *If $n \geq 3$, then*

$$v_n(1123) = \sum_{m=0}^{n/2} \left(\binom{n-m}{m} - \binom{n-2-m}{m} \right) c_{m+1}. \tag{4}$$

Proof. Let $c(y) = \sum_{n\geq 0} c_n y^n$. We may rewrite the generating function in Proposition 20 as

$$\begin{aligned}\frac{4(1-x)^2(1+x)}{(1-x+\sqrt{(1-x)^2-4x^2(1-x)})^2} &= (1+x)\left(\frac{1-\sqrt{1-4x^2/(1-x)}}{2x^2/(1-x)}\right)^2 \\ &= (1+x)c^2(x^2/(1-x)),\end{aligned}$$

where $c(y) = \frac{1-\sqrt{1-4y}}{2y} = \sum_{n\geq 0} c_n y^n$. Expanding the last expression, with use of the facts (see, e.g., [**22**, Equations 2.5.7 and 2.5.16])

$$c^j(y) = \sum_{i\geq 0} \frac{j(2i+j-1)!}{i!(i+j)!} y^i, \qquad j \geq 1,$$

and

$$\frac{1}{(1-y)^{j+1}} = \sum_{i\geq 0} \binom{i+j}{j} y^i,$$

yields

$$\begin{aligned}&(1+x)\sum_{m\geq 0} \frac{2}{m+2}\binom{2m+1}{m}\frac{x^{2m}}{(1-x)^m} \\ &\qquad = (1-x^2)\sum_{m\geq 0}\frac{2}{m+2}\binom{2m+1}{m}x^m\sum_{r\geq m}\binom{r}{m}x^r \\ &\qquad = (1-x^2)\sum_{r\geq 0}x^r\sum_{m=0}^{r}\frac{2}{m+2}\binom{2m+1}{m}\binom{r}{m}x^m.\end{aligned}$$

Extracting the coefficient of x^n, where $n \geq 3$, in the last expression yields

$$\begin{aligned}&\sum_{m=0}^{n/2}\frac{2}{m+2}\binom{2m+1}{m}\binom{n-m}{m} - \sum_{m=0}^{(n-2)/2}\frac{2}{m+2}\binom{2m+1}{m}\binom{n-2-m}{m} \\ &\qquad = \sum_{m=0}^{n/2} c_{m+1}\binom{n-m}{m} - \sum_{m=0}^{(n-2)/2} c_{m+1}\binom{n-2-m}{m},\end{aligned}$$

which gives the desired formula. $\square$

It is possible to explain directly the explicit formula (4).

Combinatorial proof of Corollary 21. For $n \geq 3$, we'll show

$$(5) \qquad v_{n,m}(1123) = \binom{m}{n-m} c_{n-m+1} - \binom{m-1}{n-m-1} c_{n-m}, \qquad n/2 \leq m \leq n.$$

Summing (5) over m, and replacing m by $n-m$ and by $n-1-m$, then gives (4). To show (5), first note that $\sigma \in V_{n,m}(1123)$ must be of the form $\sigma = 12\cdots(m-1)\sigma'$, where σ' is a 123-avoiding word of length $n-m+1$ containing letters from $[m-1]$ at most once and containing m once or twice. Conversely, any involution of the form σ belongs to $V_{n,m}(1123)$. Suppose first that m occurs once in σ'. There are then $\binom{m-1}{n-m}$ choices for the letters of $[m-1]$ appearing in σ' and c_{n-m+1} ways in which to order the letters of σ' since they must form a 123-avoiding permutation of length $n-m+1$ (see, e.g., [**13**] or [**15**]). Thus, there are $\binom{m-1}{n-m}c_{n-m+1}$ members of $V_{n,m}(1123)$ in this case.

Now suppose m occurs twice in σ'. We first show that there are $c_{i+1}-c_i$ words of length $i+1$ that avoid the pattern 123 in which each letter of $[i-1]$ appears once and the letter i appears twice, where $i \geq 1$. To see this, let $W_{i+1}(123)$ denote the set of words in question and let $S_j(123)$ denote the set of 123-avoiding permutations of length j. Given $\lambda \in W_{i+1}(123)$, let $\widetilde{\lambda}$ denote the member of $S_{i+1}(123)$ obtained by changing the first i occurring in λ to $i+1$. Then the mapping $\lambda \mapsto \widetilde{\lambda}$ is a bijection from $W_{i+1}(123)$ to the subset of $S_{i+1}(123)$ consisting of those members in which $i+1$ comes to the left of i. Note that these permutations number $c_{i+1} - c_i$, by subtraction, as there are c_i members of $S_{i+1}(123)$ in which $i+1$ comes to the right of i (note that such members must start with i, with the remaining letters constituting a member o f $S_i(123)$).

So if m occurs twice in σ', then there are $\binom{m-1}{n-m-1}$ choices for the letters in $[m-1]$ which also occur twice in σ' and, once these letters have been selected, $c_{n-m+1} - c_{n-m}$ ways in which to order them within σ'. Note that such orderings are synonymous with members of $W_{n-m+1}(123)$. Thus, there are

$$\binom{m-1}{n-m-1}(c_{n-m+1} - c_{n-m})$$

members of $V_{n,m}(1123)$ in which m occurs twice. In all, there are

$$\begin{aligned}\binom{m-1}{n-m}c_{n-m+1} &+ \binom{m-1}{n-m-1}(c_{n-m+1} - c_{n-m}) \\ &= \binom{m}{n-m}c_{n-m+1} - \binom{m-1}{n-m-1}c_{n-m}\end{aligned}$$

members of $V_{n,m}(1123)$, as desired. $\square$

We summarize the results of this section in the table below.

$\tau \backslash n$	5	6	7	8	9	10	Reference
1122	21	52	134	361	1009	2926	Proposition 18
1212,1221	21	51	127	323	835	2188	Fact 15
1123	19	43	95	217	493	1139	Corollary 21
1213, 1231	18	38	78	163	337	701	Proposition 17
1223, 1232	17	33	61	112	202	361	Proposition 16
1233	16	25	36	49	64	81	Example 5
1234	15	15	0	0	0	0	Observation 1

Table 2. Values of $v_n(\tau)$, where τ is a pattern of length four.

4. AVOIDING A PATTERN OF LENGTH FIVE

In this section, we enumerate the involutions which avoid a single pattern of length five. Our work is shortened by first noting the following.

Observation 22. *Explicit formulas and/or generating functions for the ten patterns of length five of the form* $1(\sigma+1)$ *can be derived from the results in the prior section using* (1) *and* (2).

Other results from the second section apply.

Remark 23. Since $112 \sim 121$, we have from Theorem 10 that $11223 \sim 11232$. Furthermore, since $v_n(112) = F_{n+1}$, we obtain the explicit formula

$$v_n(11223) = 1 + \sum_{r=1}^{n-1}\sum_{i=1}^{r} r\binom{r-1}{i-1}F_{n+1-r-i}\prod_{j=1}^{i-1}(n-r-j),$$

where we assume $F_m = 0$ if $m < 0$.

Remark 24. Since $112 \dot{\sim} 121$, we have from Theorem 14 that $11233 \dot{\sim} 12133$. Furthermore, since $v_{n,r}(112) = \binom{r}{n-r}$, we obtain the explicit formula

$$v_{n,r}(11233) = \sum_{m=0}^{r-1}\sum_{\ell=0}^{n-r-1}(r-m+\ell)\ell!\binom{r-m-1+\ell}{\ell}\binom{m+r+1-n+\ell}{\ell}\binom{m}{n-r-\ell-1},$$

where $1 \le r \le n-1$.

Remark 25. From Theorem 6 and Example 9, we see that the numbers $a_n = v_n(12341) = v_n(12314)$ both satisfy the recurrence

$$a_n = a_{n-1} + 3a_{n-2} + 2a_{n-4} + a_{n-5} + 5a_{n-6} + a_{n-8}, \quad n \ge 8.$$

We now consider the remaining cases for a pattern of length five.

Theorem 26. *Each of the following patterns* τ *are equivalent to one another:*

(1) 12123 (2) 12132 (3) 12213
(4) 12231 (5) 12312 (6) 12321.

Furthermore, $v_n(\tau)$ is given in all cases by

$$\sum_{m=1}^{n}\frac{1}{m}\left(\sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^{m+j-n}2^{m-j}\binom{m}{j}\binom{j}{i}\binom{m+i+j}{m-1}\binom{j-i}{m+j-n}\right),\quad n\geq 1.$$

Proof. To prove the first statement, we consider more general results concerning set partitions. Let $P_n(\sigma)$ denote the set of partitions of $[n]$ which avoid the pattern σ, where partitions are represented canonically. Two patterns σ and τ are said to be *strongly partition-equivalent* if for each n there exists a bijection from $P_n(\sigma)$ to $P_n(\tau)$ which preserves the size of the i-th block for each i. From [**11**, Fact 4.8] (see also [**9**, Theorem 48]), all the patterns of a given size k that start with 12 and that contain two occurrences of the symbol 1 and one occurrence of the symbol 3, with each of their remaining symbols equal to 2, are mutually strongly partition-equivalent. Particularizing these results to the case when $k = 5$ and involutions implies the equivalence of the patterns listed above.

To complete the proof, we determine an explicit formula for $v_n(\tau)$ in the case $\tau = 12312$. Define $m(\pi)$ to be the maximal letter of π. By induction on s, one may verify that π belongs to $V_n(12312)$ if and only if $\pi = 1\pi^{(1)}$ or there exists $s \geq 1$ such that

$$\pi = 1\pi^{(1)}1\pi^{(2)}m(\pi^{(1)})\pi^{(3)}m(\pi^{(2)})\cdots\pi^{(s)}m(\pi^{(s-1)})\pi^{(s+1)},$$

where any letter of $\pi^{(j)}$ is greater than any letter of $\pi^{(j-1)}$ for $j = 1, 2, \ldots, s+1$ (we define $\pi^{(0)} = 1$), with each $\pi^{(j)}$ avoiding 12312. Let $f(x)$ be the generating function for the number of 12312-avoiding involutions of length n, and let $g(x)$ be the generating function for the number of 12312-avoiding involutions π of length n where the letter $m(\pi)$ appears exactly once in π. From the above decomposition, we have

$$f(x) = 1 + xf(x) + \frac{x^2f^2(x)}{1-xg(x)},$$
$$g(x) = x + xg(x) + \frac{x^2g(x)(f(x)+1)}{1-xg(x)}.$$

Solving for the quantity $\frac{x^2}{1-xg(x)}$ in these two equations, and equating results, gives

$$\frac{(1-x)f(x)-1}{f^2(x)} = \frac{(1-x)g(x)-x}{g(x)(f(x)+1)},$$

which reduces to $(xf(x)+1)g(x) = xf^2(x)$. Now substituting into this last equation $g(x) = \frac{1}{x} - \frac{xf^2(x)}{(1-x)f(x)-1}$, and simplifying, implies that the generating function $f(x)$ satisfies

$$f(x) = 1 + 2xf(x) + x(x-1)f^2(x) + x^2f^3(x).$$

Let $h(x)=f(x)-1$, so that

$$h(x)=x(h(x)+1)(2+(x-1)(h(x)+1)+x(h(x)+1)^2).$$

By the Lagrange inversion formula, we have

$$\begin{aligned}
h(x)&=\sum_{m\geq1}\frac{[y^{m-1}]}{m}\left(x^m(y+1)^m(2+(x-1)(y+1)+x(y+1)^2)^m\right)\\
&=\sum_{m\geq1}\frac{[y^{m-1}]}{m}\left(\sum_{j=0}^{m}\sum_{i=0}^{j}\binom{m}{j}\binom{j}{i}2^{m-j}(y+1)^{m+j+i}(x-1)^{j-i}x^{m+i}\right)\\
&=\sum_{m\geq1}\frac{1}{m}\left(\sum_{j=0}^{m}\sum_{i=0}^{j}\binom{m}{j}\binom{j}{i}\binom{m+i+j}{m-1}2^{m-j}(x-1)^{j-i}x^{m+i}\right).
\end{aligned}$$

Extracting the coefficient of x^n in $h(x)$ completes the proof. □

The cases left now are 12313, 12331, 11234, and 12134, the last two of which we were unable to find explicit expressions for $v_n(\tau)$ or its generating function. The remainder of this section is then devoted to the cases 12313 and 12331.

Proposition 27. *If $n\geq 0$, then*

$$\begin{aligned}
v_n(12313)=1&+\sum_{m=1}^{n/2}\sum_{j=1}^{m}\sum_{b=0}^{n}\frac{(-1)^b}{m}\binom{m}{j}\binom{2m+j}{m-1}\binom{n-b}{2m+2j}\binom{j-1+b}{b}\\
&+\sum_{m=1}^{n/2}\frac{1}{m+1}\binom{2m}{m}\binom{n}{2m}.
\end{aligned}$$

Proof. Let $f=f(x)=\sum_{n\geq0}v_n(12313)x^n$. We first write an equation for f. Let $\pi\in V_n(12313)$, where $n\geq1$. Then π must have one of the following three forms: (i) 1α, where α does not contain 1; (ii) $1\alpha1\beta$, where neither α nor β contains 1 and any letter of α is less than any letter of β; or (iii) $1\alpha1\beta$, where neither α nor β contains 1 and α and β have a letter in common. Combining the three cases above implies

$$f=1+xf+x^2f^2+g,$$

where $g=g(x)$ is the generating function that counts involutions of length n of the form (iii).

Let us now write an equation for g. Suppose that α and β share the letter r in the third case above. If $r>2$, then we would have the subsequence $12r1r$ in π, which is an occurrence of 12313, so $r=2$. Thus $\alpha=2\alpha'$ and $\beta=\beta'2\beta''$ in (iii), with α', β' and β'' each not containing 1 or 2. If α' is empty, then we have

$\pi = 121\beta'2\beta''$, and considering whether or not β' and β'' share a letter gives a contribution of $x^2g + x^4f^2$ in this case. If α' is non-empty, then $\pi = 12\alpha'1\beta'2\beta''$ and π avoiding 12313 implies α', β' and β'' are mutually disjoint, which gives a contribution of $x^4f^2(f-1)$ in this case. Thus, we have $g = x^2g+x^4f^2+x^4f^2(f-1)$, or $g = \frac{x^4}{1-x^2}f^3$, which implies

$$f = 1 + xf + x^2f^2 + \frac{x^4}{1-x^2}f^3.$$

We rewrite this last equality as

$$h = \frac{x^2}{1-x}\left(h+\frac{1}{1-x}\right)^2\left(1+\frac{x^2}{1-x^2}\left(h+\frac{1}{1-x}\right)\right),$$

where $h = h(x) = f(x) - \frac{1}{1-x}$. By the Lagrange inversion formula, we have

$$\begin{aligned}
h(x) &= \sum_{m\geq 1}\frac{x^{2m}}{m(1-x)^m}[y^{m-1}]\left[\left(y+\frac{1}{1-x}\right)^{2m}\left(1+\frac{x^2}{1-x^2}\left(y+\frac{1}{1-x}\right)\right)^m\right]\\
&= \sum_{m\geq 1}\sum_{j=0}^{m}\frac{x^{2m+2j}}{m(1-x)^m(1-x^2)^j}\binom{m}{j}[y^{m-1}]\left[\left(y+\frac{1}{1-x}\right)^{2m+j}\right]\\
&= \sum_{m\geq 1}\sum_{j=0}^{m}\frac{x^{2m+2j}}{m(1-x)^{2m+j+1}(1-x^2)^j}\binom{m}{j}\binom{2m+j}{m-1}\\
&= \sum_{m\geq 1}\sum_{j=0}^{m}\frac{x^{2m+2j}}{m(1-x)^{2m+2j+1}(1+x)^j}\binom{m}{j}\binom{2m+j}{m-1}\\
&= \sum_{m\geq 1}\sum_{j=1}^{m}\sum_{a,b\geq 0}(-1)^b\frac{x^{2m+2j+a+b}}{m}\binom{m}{j}\binom{2m+j}{m-1}\binom{2m+2j+a}{a}\binom{j-1+b}{b}\\
&\quad+\sum_{m\geq 1}\sum_{a\geq 0}\frac{x^{2m+a}}{m}\binom{2m}{m-1}\binom{2m+a}{a}.
\end{aligned}$$

Thus, the coefficient of x^n in h is given by

$$\sum_{m=1}^{n/2}\sum_{j=1}^{m}\sum_{b=0}^{n}\frac{(-1)^b}{m}\binom{m}{j}\binom{2m+j}{m-1}\binom{n-b}{2m+2j}\binom{j-1+b}{b} + \sum_{m=1}^{n/2}\frac{1}{m+1}\binom{2m}{m}\binom{n}{2m}.$$

Noting $f(x) = h(x) + \frac{1}{1-x}$ completes the proof. $\square$

We now consider the case 12331. Given $n \geq 1$ and $1 \leq k \leq n-1$, let $a_{n,k}$ count the members $\pi = \pi_1\pi_2 \cdots \pi_n \in V_n(12331)$ containing a single 1 of the form $\pi = 12 \cdots kj\pi'$ for some $j \in [k]$. Let $b_{n,k}$ count the members of $V_n(12331)$ having this same form, but instead containing two 1's. For example, we have $a_{5,3} = 4$, the enumerated involutions being 12323, 12324, 12332, 12334, and $b_{6,4} = 5$, the involutions being $12341i$, where $i \in [2,5]$, and 123421. Furthermore, let us take $a_{n,0} = \delta_{n,0}$, $a_{n,n} = 1$, and $b_{n,0} = b_{n,n} = 0$ for all $n \geq 0$ (note $a_{n,n} = 1$ counts the single involution $12 \cdots n$).

Lemma 28. *The arrays $a_{n,k}$ and $b_{n,k}$ can assume non-zero values only when $0 \leq k \leq n$ and satisfy the recurrences*

$$a_{n,k} = a_{n-1,k-1} + b_{n-1,k-1}, \qquad 1 \leq k \leq n, \tag{6}$$

and

$$b_{n,k} = \sum_{i=k-1}^{n-2}(a_{n-2,i} + b_{n-2,i}) + \sum_{i=k-1}^{n-3}\sum_{j=i-1}^{n-4}(a_{n-4,j} + b_{n-4,j}), \qquad 2 \leq k \leq n-1. \tag{7}$$

If $n \geq 0$, then $a_{n,0} = \delta_{n,0}$ and $b_{n,n} = b_{n,0} = 0$, with $b_{n,1} = \sum_{k=0}^{n-2}(a_{n-2,k} + b_{n-2,k})$ for $n \geq 2$.

Proof. The initial values are easily verified. Note that $b_{n,1} = \sum_{k=0}^{n-2}(a_{n-2,k} + b_{n-2,k})$ if $n \geq 2$, which is seen upon deleting both letters 1 at the beginning. Removing the 1 from an involution $\pi = \pi_1\pi_2 \cdots \pi_n$ enumerated by $a_{n,k}$ yields either an involution counted by $b_{n-1,k-1}$ or by $a_{n-1,k-1}$, depending on whether or not the letter 2 is repeated, which gives (6).

To show (7), first note that there are $\sum_{i=k-1}^{n-2}(a_{n-2,i} + b_{n-2,i})$ members of $V_n(12331)$ enumerated by $b_{n,k}$, where $2 \leq k \leq n-1$, of the form $12 \cdots k1\pi'$, upon deleting both occurrences of 1 and considering whether or not the resulting involution $2 \cdots k\pi'$ has two occurrences of 2. Otherwise, we have $\pi = 12 \cdots kr(k+1)(k+2) \cdots (i+1)1\pi'$ for some $r \in [2,k]$ and $i \in [k-1, n-3]$. Note that $r = 2$, for if $r > 2$, then we would have the subsequence $12rr1$, which is an occurrence of 12331. Deletion of both 1's and both 2's then results in involutions of the form $34 \cdots (i+1)\pi'$, which are counted by $\sum_{j=i-1}^{n-4}(a_{n-4,j} + b_{n-4,j})$, upon considering whether or not the letter 3 is repeated. Summing this over i gives (7) and completes the proof. □

If $n \geq 0$, then let $a_n = \sum_{k=0}^{n} a_{n,k}$ and $b_n = \sum_{k=0}^{n} b_{n,k}$. From the definitions, we have $v_n(12331) = a_n + b_n$ for all n.

The function

$$g(v,t) = 1 - vt + \frac{v^2t^2}{1-v} - \frac{v^4t^4}{(1-v)^2}$$

is encountered in our derivation below of the generating function for the numbers $v_n(12331)$. Let us make some preliminary observations concerning it. Upon multiplying through by $(1-v)^2$, we see that the equation $g(v,t) = 0$ has one negative root and either one or three positive roots in v where $t \neq 0$ is fixed, by Descartes' rule of signs. In fact, for all t sufficiently close to zero, we see that $g(v,t)$ has roots in both the intervals $\left(\frac{1}{2}, 1\right)$ and $\left(1, \frac{3}{2}\right)$, by the intermediate value theorem. Let us denote these roots by $v_\pm(t)$. Using the gfun package, it is possible to find formal power series in t for $v_\pm(t)$, with the other roots of $g(v,t) = 0$ not admitting to such an expansion. The first few terms of these series are

$$\begin{aligned}
v_\pm(t) &= 1 + at^2 + \frac{1}{5}(1+3a)t^3 + \frac{1}{5^2}(56+63a)t^4 + \frac{1}{5^3}(407+561a)t^5 \\
&\quad + \frac{1}{5^3}(1198+2104a)t^6 + \frac{1}{5^5}(71889+121847a)t^7 \\
&\quad + \frac{1}{5^5}(1159237+1853401a)t^8 + \frac{1}{5^7}(15647267+25033591a)t^9 \\
&\quad + \frac{1}{5^8}(232895389+377177972a)t^{10} + \frac{1}{5^7}(134073521+217477958a)t^{11} \\
&\quad + \frac{1}{5^{10}}(50472627212+81681698701a)t^{12} \\
&\quad + \frac{1}{5^{11}}(751784647481+1215815470663a)t^{13} \\
&\quad + \frac{1}{5^{12}}(11401941720791+18446604839668a)t^{14} \\
&\quad + \frac{1}{5^{13}}(172999678091967+279943026628566a)t^{15} + O(t^{16}),
\end{aligned}$$

where $a = \dfrac{1 \pm \sqrt{5}}{2}$.

Using $v_\pm(t)$, one may provide an expression for the generating function for the numbers $v_n(12331)$.

Theorem 29. *Let* $f(t) = \sum_{n \geq 0} v_n(12331)t^n$. *Then we have*

$$f(t) = \frac{(1-v_+)(1-v_-)(v_+v_-(v_++v_-) - v_+^2 - v_+v_- - v_-^2)}{v_+^2v_-^2t^3\left[(1-v_+)(1-v_-) + v_+v_-t^2)\right]} - \frac{1}{t},$$

where $v_\pm = v_\pm(t)$ *are as given above.*

Proof. We determine the generating function for the sequence $v_n(12331) = a_n + b_n$, using the recurrences in the prior lemma. First note that by (6), relation (7) is equivalent to

$$b_{n,k} = \sum_{i=k}^{n-1} a_{n-1,i} + \sum_{i=k-1}^{n-3}\sum_{j=i}^{n-3} a_{n-3,j}, \qquad 2 \le k \le n-1. \tag{8}$$

Define $A_n(v) = \sum_{i=0}^{n} a_{n,i}v^i$ and $B_n(v) = \sum_{i=0}^{n} b_{n,i}v^i$; note that $A_n(1) = a_n$ and $B_n(1) = b_n$. Multiplying (6) by v^k, summing over $k = 1, 2, \dots, n$ and noting $a_{n,0} = 0$ if $n \ge 1$, we obtain

$$A_n(v) = vA_{n-1}(v) + vB_{n-1}(v).$$

Multiplying (8) by v^k, summing over $k = 2, 3, \dots, n-1$, and noting $a_{n,0} = b_{n,0} = 0$ if $n \ge 1$ and $b_{n,1} = A_{n-2}(1) + B_{n-2}(1) = A_{n-1}(1)$ if $n \ge 2$, we obtain

$$\begin{aligned}
B_n(v) - vA_{n-1}(1) &= \sum_{k=2}^{n-1} v^{k-1} \sum_{i=k}^{n-1} a_{n-1,i} + \sum_{k=2}^{n-1} v^k \sum_{i=k-1}^{n-3}\sum_{j=i}^{n-3} a_{n-3,j} \\
&= \sum_{k=1}^{n-1} \frac{v^2 - v^{k+1}}{1-v} a_{n-1,k} + \sum_{k=1}^{n-3} (kv^2 + (k-1)v^3 + \cdots + v^{k+1}) a_{n-3,k} \\
&= \frac{v}{1-v}(vA_{n-1}(1) - A_{n-1}(v)) + \frac{v^3}{(1-v)^2}(A_{n-3}(v) - A_{n-3}(1)) + \frac{v^2}{1-v} h_{n-3},
\end{aligned}$$

where $h_{n-3} = \frac{\partial}{\partial v} A_{n-3}(v)\Big|_{v=1}$. Thus for all $n \ge 2$, we have

$$\begin{aligned}
A_n(v) &= vA_{n-1}(v) + vB_{n-1}(v) \\
B_n(v) &= \frac{v}{1-v}(A_{n-1}(1) - A_{n-1}(v)) + \frac{v^3}{(1-v)^2}(A_{n-3}(v) - A_{n-3}(1)) + \frac{v^2}{1-v} h_{n-3},
\end{aligned}$$

with $A_0(v) = 1$, $A_1(v) = v$ and $B_0(v) = B_1(v) = 0$ (we define $A_{-1}(v) = 0$).

Let $H(t) = \sum_{n\ge0} h_n t^n$, $A(t,v) = \sum_{n\ge0} A_n(v)t^n$ and $B(t,v) = \sum_{n\ge0} B_n(v)t^n$. Multiplying the last two recurrences by t^n and summing over $n \ge 2$ yields

$$A(t,v) - vt - 1 = vt(A(t,v) - 1) + vtB(t,v) \tag{9}$$

and

$$B(t,v) = \frac{vt}{1-v}(A(t,1) - A(t,v)) + \frac{v^3t^3}{(1-v)^2}(A(t,v) - A(t,1)) + \frac{v^2t^3}{1-v}H(t).$$

Hence,

$$\left(1 - vt + \frac{v^2t^2}{1-v} - \frac{v^4t^4}{(1-v)^2}\right) A(t,v) \tag{10}$$

$$= 1 + \frac{v^2t^2}{1-v}\left(1 - \frac{v^2t^2}{1-v}\right)A(t,1) + \frac{v^3t^4}{1-v}H(t).$$

From the preceding discussion, we see that the kernel $1 - vt + \frac{v^2t^2}{1-v} - \frac{v^4t^4}{(1-v)^2}$ of the functional equation (10) has two roots $v_\pm(t)$ which are expressible as formal power series in t. By substituting $v = v_+ = v_+(t)$ and $v = v_- = v_-(t)$ into (10), we obtain

$$0 = 1 + \frac{v_+^2t^2}{1-v_+}\left(1 - \frac{v_+^2t^2}{1-v_+}\right)A(t,1) + \frac{v_+^3t^4}{1-v_+}H(t),$$
$$0 = 1 + \frac{v_-^2t^2}{1-v_-}\left(1 - \frac{v_-^2t^2}{1-v_-}\right)A(t,1) + \frac{v_-^3t^4}{1-v_-}H(t),$$

which implies

$$v_+^2v_-^2t^2\left[v_-\left(1 - \frac{v_+^2t^2}{1-v_+}\right) - v_+\left(1 - \frac{v_-^2t^2}{1-v_-}\right)\right]A(t,1) = v_+^3(1-v_-) - v_-^3(1-v_+).$$

Hence,

$$A(t,1) = \frac{(1-v_+)(1-v_-)(v_+v_-(v_++v_-) - v_+^2 - v_+v_- - v_-^2)}{v_+^2v_-^2t^2\left[(1-v_+)(1-v_-) + v_+v_-t^2)\right]}.$$

By (9), we have $A(t,1)+B(t,1) = \frac{1}{t}(A(t,1)-1)$, which implies that the generating function $f(t) = \sum_{n\geq 0} v_n(12331)t^n = \sum_{n\geq 0}(a_n+b_n)t^n$ is given by

$$\frac{(1-v_+)(1-v_-)(v_+v_-(v_++v_-) - v_+^2 - v_+v_- - v_-^2)}{v_+^2v_-^2t^3\left[(1-v_+)(1-v_-) + v_+v_-t^2)\right]} - \frac{1}{t},$$

as desired. □

It is possible to give more explicit expressions for the functions $v_\pm$ appearing above.

Let

$$m_1 = \sqrt{-5 + 4t + 10t^2 + 54t^3 + 139t^4 + 82t^5 + 279t^6},$$
$$m_2 = \sqrt[3]{24t + 24t^2 - 52t^3 + 96t^4 - 156t^5 - 188t^6 - 8 + 12\sqrt{3}m_1t^3},$$
$$m_3 = \sqrt{\frac{8t^2(1-2t-3t^2+t^3-11t^4)}{m_2} + 2t^2m_2 + (3+8t^2)(1+t)^2}.$$

Corollary 30. *The generating function $f(t)$ is algebraic. In fact if*

$$h(t) = \frac{18\sqrt{3}((1+t)^3(1+4t^2) + 8(2+t)t^5)}{m_3} - \frac{24t^2(1-2t-3t^2+t^3-11t^4)}{m_2}$$

and

$$g(t) = \frac{\sqrt{h(t) + 6(3+8t^2)(1+t)^2} - 3 - 3t - \sqrt{3}m_3}{12t^3},$$

then $v_+(t)$ *is* $g(t)$ *with the branch* $m_1^2 < 0$ *and* $v_-(t)$ *is* $g(t)$ *with the branch* $m_1^2 > 0$ *and* $m_2^3 < 0$.

Proof. The first statement follows from Theorem 29 since $f(t)$ is a rational function of v_+ and v_-, which are both seen to be algebraic. Note that $v_+(t)$ and $v_-(t)$ are zeros of the kernel in (10). By using Maple (and simplifying somewhat the expressions that result), we obtain the given formulas for v_+ and v_-. □

Using the power series of $v_\pm$, one may find, with the help of Maple, the first few terms of the generating function $f(t)$:

$$\begin{aligned} f(t) &= 1 + t + 2t^2 + 4t^3 + 10t^4 + 25t^5 + 67t^6 + 182t^7 + 512t^8 + 1460t^9 + 4241t^{10} \\ &\quad + 12453t^{11} + 36999t^{12} + 110865t^{13} + 334929t^{14} + 1018545t^{15} + O(t^{16}). \end{aligned}$$

We summarize the results of this section in the table below.

$\tau\backslash n$	6	7	8	9	10	11	Reference
11223, 11232	69	193	572	1730	5452	17573	Remark 23
11233, 12133	67	183	520	1516	4562	14097	Remark 24
12123, 12132 12213, 12231 12312, 12321	68	187	534	1544	4554	13576	Theorem 26
12233	66	178	498	1433	4258	13016	Observation 22 and Proposition 18
12331	67	182	512	1460	4241	12453	Theorem 29
12313	67	181	503	1414	4037	11642	Proposition 27
12323, 12332	66	177	484	1339	3742	10538	Observation 22 and Fact 15
11234	66	173	473	1290	3623	10193	Open
12134	65	164	428	1104	2904	7607	Open
12234	64	157	396	977	2446	6069	Observation 22 and Corollary 21
12314, 12341	64	154	381	924	2272	5545	Remark 25
12324, 12342	63	146	344	787	1804	4071	Observation 22 and Proposition 17
12334, 12343	62	135	292	600	1210	2381	Observation 22 and Proposition 16
12344	61	121	211	337	505	721	Example 5
12345	60	105	105	0	0	0	Observation 1

Table 3. Values of $v_n(\tau)$, where τ is a pattern of length five.

5. CONCLUSION

Making use of both algebraic and combinatorial methods, we have provided explicit expressions for $v_n(\tau)$ and/or its generating function when τ is a pattern of length at most five with two exceptions. In some cases, we have found formulas for $v_{n,k}(\tau)$ as well. For the particular patterns 1123 and 12331, we used the kernel method to solve the functional equations that arise. In the case of 1123, the kernel method gives a quadratic equation, and there is thus a closed form which may then be explained by combinatorial reasoning. In the case of 12331, however, the equation we get is quartic and no compact closed form for $v_n(12331)$ seems possible, though one may express the generating function in terms of certain formal power series involving two of the real roots of this quartic. Still unresolved are the cases 11234 and 12134. We also have shown several general avoidance results concerning partial matchings, including the fact that the generating function $\sum_{n\geq 0} v_n(12\cdots ki)x^n$ is rational for all k and i, with $12\cdots ki \sim 12\cdots ik$. In this direction, we still seek explicit formulas for generating functions of patterns such as $12\cdots k12$ and $12\cdots k13$, which are seen not to be rational in general. We also seek pattern equivalences for partial matchings which extend the one in Theorem 6.

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