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LIE SYMMETRIES AND NOETHER SYMMETRIES

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We demonstrate that so-called nonnoetherian symmetries with which a known first integral is associated of a differential equation derived from a Lagrangian are in fact noetherian. The source of the misunderstanding lies in the nonuniqueness of the Lagrangian.

1. INTRODUCTION

In 1918 Noether [**24**] published a fundamental paper on the relationship between the existence of a symmetry of the Action Integral and the existence of a corresponding first integral when the variational principle of Hamilton was applied to produce the Euler-Lagrange equation. The symmetry was a consequence of the invariance of the Action Integral,

$$A = \int_{t_0}^{t_1} L(t,\, q,\, \dot{q},\, \ldots)\mathrm{d}t, \tag{1}$$

under an infinitesimal transformation of the independent variable and the dependent variables generated by the differential operator

$$\Gamma = \tau(t,\, q,\, \dot{q},\, \ldots)\partial_t + \eta_i(t,\, q,\, \dot{q},\, \ldots)\partial_{q_i}, \tag{2}$$

where the order of the derivatives of the dependent variables in the coefficient functions was less than or equal to the order of the derivatives occurring in the Lagrangian. In the present article we restrict ourselves to one independent variable, but q may denote many dependent variables. Naturally Noether [**24**] allowed for

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t to be a multivariable thereby enabling the theorem to apply to Lagrangian field theories. The corresponding first integral is given by

$$I = f(t,\, q,\, \dot{q},\, \ldots) - \left[\tau L + (\eta_i - \dot{q}_i \tau)\,\frac{\partial L}{\partial \dot{q}_i} - \ldots\right], \tag{3}$$

where the function $f(t,\, q,\, \dot{q},\, \ldots)$ is called the boundary function since its origin is in the contribution to the variation of the Action Integral due to the infinitesimal variation of the endpoints of the integral because of the transformation in the independent variable. We remind the reader that the invariance of the Action Integral under the infinitesimal transformation generated by Γ has nothing to do with the application of Hamilton's Principle which specifies zero end-point variations.

Over the intervening years Noether's Theorem has become the stuff of textbooks. Unfortunately it is generally misrepresented (for a modest sampling see [**2, 18, 3**]) as applying to point symmetries only and in a mechanical context to Lagrangians of the form $L = T - V$ which, as everyone should know, is a rather specialised and restricted form, certainly in the context of the Calculus of Variations to which Noether's Theorem refers. Despite the thorough review by Sarlet and Cantrijn [**29**] of 1981 we find a later writer [**3**] claiming the authority of Courant and Hilbert [**2**] the further to propagate the bowdlerised form of the theorem. Since Hilbert was present at the Festschrift in honour of Felix Klein when Noether presented her theorem, it is surprising that the presentation in the text is incomplete until one learns that Hilbert's contribution was his name.

If one uses the point form of (2) to determine the symmetries of (1), be the Lagrangian of the specific form above or more general, it is not surprising that the Theorem can fail to produce integrals, such as the Laplace-Runge-Lenz vector, well known from other, often elementary, considerations. This then becomes 'The Failure of Noether's Theorem'! This failure led later workers diligently to seek new methods to overcome the deficiencies of the theorem. A notable example of this is to be found in the s- and g-symmetries proposed by Hojman and his collaborators [**4, 5, 6, 7, 8**]. Later it was demonstrated [**15**] that the first integrals for the examples considered in their papers could easily be obtained by a proper application of Noether's Theorem. A more recent illustration is to be found in the paper by Luo and Jia [**19**] in their determination of an integral for a problem in relativistic mechanics which, again, could not be obtained by an application of Noether's Theorem. Their problem possessed a Lagrangian of the form above, albeit decorated with asterisks which, we suspect, had no great bearing on their analysis. The Lie symmetries used to determine the integral were termed nonnoetherian. Some point was made of the fact that the symmetry analysis was recast in terms of the canonically conjugate variables of the corresponding Hamiltonian. This was not a particularly original approach (see [**27**]).

It is the central theme of this work to demonstrate, by means greatly elementary, how allegedly nonnoetherian Lie symmetries can in fact be recast as noetherian symmetries provided that one goes to the trouble to read Noether's original work. It is well known that the first integrals of a first-order Lagrangian can be obtained from a single Lagrangian – usually the 'standard' Lagrangian – by the use of generalised, or velocity-dependent, symmetries. It is not the point of this paper to do this. Rather it is to demonstrate that to all Lie point symmetries which lead to a nonzero Jacobi Last Multiplier there corresponds a Lagrangian for which those symmetries are Noetherian and so can be used to construct first integrals according to the prescription of Noether's Theorem. In a more sophisticated elaboration we demonstrate the direct connection between our results simply obtained and the fundamental relationship between Jacobi's Last Multiplier [**9, 10, 11, 12, 13**] and a Lagrangian and the connection between the Last Multiplier and Lie symmetries demonstrated so long ago by Lie [**16**].

In §2 we provide an analysis in terms of the Lagrangian for the free particle. This enables a ready appreciation of the fact which we wish to emphasise which is that Noether's Theorem necessarily relates a symmetry and its integral to a Lagrangian. In §3 we relate the results of the previous section to the relationship between Lie symmetries and a Lagrangian through the Last Multiplier of Jacobi. We provide a brief Discussion in §4.

2. THE FREE PARTICLE: THE MYTH OF NONNOETHERIAN SYMMETRIES

The free particle has the equation of motion

$$\ddot{q} = 0 \tag{4}$$

with the Lie point symmetries and associated first integrals (obtained using the standard method for determining a first integral using a Lie point symmetry)

Symmetry	First Integral
$\Gamma_1 = \partial_q$	$I_1 = \dot{q}$
$\Gamma_2 = t\partial_q$	$I_2 = t\dot{q} - q$
$\Gamma_3 = \partial_t$	$I_3 = \dot{q}$
$\Gamma_4 = 2t\partial_t + q\partial_q$	$I_4 = \dot{q}\,(t\dot{q} - q)$
$\Gamma_5 = t^2\partial_t + tq\partial_q$	$I_5 = t\dot{q} - q$
$\Gamma_6 = q\partial_q$	$I_6 = \dfrac{t\dot{q} - q}{\dot{q}}$
$\Gamma_7 = q\partial_t$	$I_7 = \dfrac{t\dot{q} - q}{\dot{q}}$
$\Gamma_8 = qt\partial_t + q^2\partial_q$	$I_8 = \dfrac{t\dot{q} - q}{\dot{q}}.$

The free particle has the standard Lagrangian

$$L = \frac{1}{2}\dot{q}^2. \tag{5}$$

The Noether point symmetries of the corresponding Action Integral and their associated first integrals are

Symmetry	Boundary term	First Integral
$\Gamma N_1 = \partial_q$	$f_1 = 0$	$IN_1 = -\dot{q}$
$\Gamma N_2 = t\partial_q$	$f_2 = q$	$IN_2 = q - t\dot{q}$
$\Gamma N_3 = \partial_t$	$f_3 = 0$	$IN_3 = -\frac{1}{2}\dot{q}^2$
$\Gamma N_4 = t\partial_t + q\partial_q$	$f_4 = 0$	$IN_4 = \dot{q}\,(q - t\dot{q})$
$\Gamma N_5 = t^2\partial_t + 2tq\partial_q$	$f_5 = q^2$	$IN_5 = (q - t\dot{q})^2$

Apart from some differences in sign and functional differences in the expression of the integral one notes several differences. (Recall that, if I is an integral, then so is any function of I. In the case of Lie symmetries one tends to omit the possibility of an arbitrary function of I. On the other hand there is no ambiguity in the functional expression given by Noether's Theorem.) The integrals corresponding to Γ_1 and Γ_2 coincide with the integrals for the corresponding Noether symmetries. These are the symmetries known as the solution symmetries since the coefficients of ∂_q are the solutions of the differential equation under consideration, (4). In the case of the following three symmetries, each triplet being a representation of $\mathrm{sl}(2,R)$, there are differences in the functional representation of the integral. However, the noticeable feature is the absence of any symmetry, indeed integral, corresponding to Γ_6, Γ_7 and Γ_8 in the set of Noether point symmetries. Such symmetries have been termed nonnoetherian.

The symmetries, Γ_6, Γ_7 and Γ_8, constitute a subalgebra of $\mathrm{sl}(3,R)$ which is the algebra [**17**] [p 405] (see also [**1**]) of (4). The algebra itself is $A_{3,3} \Leftrightarrow A_1 \oplus_s 2A_1$ also known as $D \oplus_s 2A_1$, which represents the algebra of translations in the plane and dilation, in the Mubarakzyanov Classification Scheme [**20, 21, 22, 23**]. As this same subalgebra is found with other combinations of the elements of $\mathrm{sl}(3,R)$ which do correspond to Noether point symmetries for the Lagrangian, (5) and the different representations of this subalgebra can be transformed from one to the other by means of a point transformation, one can only find it a little strange that one representation works for the Action Integral (5) and the other does not.

The resolution of this conundrum is suggested by the implication of the last sentence in the paragraph above. One can transform one representation of the subalgebra to another representation and this leaves (4) invariant. However, this does not imply that the Lagrangian in the Action Integral is going to be unchanged. The relationship between first integral, boundary term, symmetry and Lagrangian is given in (3). For the three 'nonnoetherian' Lie point symmetries of (4) we have a corresponding integral. Consequently we can solve (3) for L provided that we make an assumption that $f = 0$ in each case. We illustrate the method of construction in the case of Γ_6. When we make the appropriate substitutions into (3), we have the partial differential equation

$$q\frac{\partial L}{\partial \dot{q}} = -\frac{t\dot{q} - q}{\dot{q}} \tag{6}$$

which is easily integrated to give

$$L = \log \dot{q} - \frac{t\dot{q}}{q} + g(t,\, q), \tag{7}$$

where g is an arbitrary function of integration in the indicated arguments. We substitute (7) into the Euler-Lagrange equation and make use of (4) to find that

$$\frac{\partial g}{\partial q} = -\frac{1}{q} \quad\Longrightarrow\quad g = -\log q$$

to within an ignorable function of time. The calculations for Γ_7 and Γ_8 proceed in a similar fashion. We find that that the Lagrangian corresponding to each of the allegedly nonnoetherian symmetries is, respectively,

$$\Gamma_6: \qquad L_6 = \log\frac{\dot{q}}{q} - \frac{t\dot{q}}{q} + h(t) \tag{8}$$

$$\Gamma_7: \qquad L_7 = \frac{(t\dot{q}-q)^2}{2q^2\dot{q}} + \dot{q}h(q) \tag{9}$$

$$\Gamma_8: \qquad L_8 = (t\dot{q}-q)\left\{\frac{h(q)}{t^2} + \frac{1}{q^2}\left[\log\left(t\dot{q}-q\right) - \log\left(\dot{q}\right)\right]\right\} \tag{10}$$

in each of which $h(\cdot)$ is an arbitrary function of its indicated argument. In (8) and (9) the function may be taken to be zero without loss of generality since in both cases it can be written as a total derivative with respect to time. In the case of (10) the linearity in $\dot{q}$ of the coefficient of $h(q)$ means that the function can be set at zero because it cannot contribute to the Euler–Lagrange Equation. We note that the Jacobi Last Multipliers in each case can be obtained using Lie's matrix using the pairs of symmetries $\{\Gamma_1,\, \Gamma_7\}$, $\{\Gamma_7,\, \Gamma_4\}$ and $\{\Gamma_3,\, \Gamma_8\}$, respectively.

So much for nonnoetherian Lie symmetries in this example!

3. THE LIE POINT SYMMETRIES OF THE FREE PARTICLE, THE LAST MULTIPLIER OF JACOBI AND THE CORRESPONDING LAGRANGIANS

In the second quarter of the nineteenth century Jacobi developed a theory of a last multiplier, now called the Jacobi Last Multiplier, for systems of differential equations [**9, 10, 11, 12, 13**]. Since the existence of multipliers and of first integrals is intimately related, in hindsight one cannot be surprised that Lie was able to relate the calculation of a Jacobi Last Multiplier to the existence of Lie symmetries of the system under consideration [**16**]. Here we use the definition of a first integral as a function of the independent and dependent variables and their derivatives which cannot be eliminated by the differential equations under consideration with a zero gradient when the differential equations are taken into account. Naturally in the case of a single independent variable the gradient is simply the total derivative with respect to the independent variable. When one takes the determinant of a matrix consisting of the vector field of the system – necessarily first-order –of differential equations and a number of symmetries to make the matrix square, the reciprocal of the determinant is a Jacobi Last Multiplier. If two multipliers, M_1 and M_2, are known, their ratio is a first integral. Furthermore a Jacobi Last Multiplier and the corresponding Lagrangian are related according to [**13, 31**]

$$\frac{\partial^2 L}{\partial \dot{q}^2} = M \tag{11}$$

for a one-degree-of-freedom system. A partial answer to the problem in more than one dimension has been provided in [**31, 28, 25**]. Consequently we see that there is a connection between the different facets of the given system with which we deal. We have a differential equation possessing symmetry. We write everything in terms of equations of the first order and write the symmetries in their first extended form and construct the matrix. In the case of a nonzero determinant the inverse gives the Last Multiplier. If it is possible to determine more than one multiplier, the ratio gives an integral. For each multiplier we can use (11) to determine the dependence of the Lagrangian on $\dot{q}$ and hence the complete expression using the same method as in §2. If it so happens that a Lagrangian of the form $\frac{1}{2}\dot{q}^2 + \dot{q}g(t,q) + h(t,q)$ is known, it follows from (11) that one of the multipliers is a constant. Consequently every other multiplier is a first integral.

We illustrate these ideas with the same free particle as we used in §2. For the time being we assume that we are unaware of any first integrals. The only information at our disposal is the equation of motion and the associated Lie point symmetries. We write (4) as the system

$$\dot{q} = p \tag{12}$$

$$\dot{p} = 0. \tag{13}$$

Since the vector field contains three components, we use the symmetries listed above in pairs so that there are 28 determinants to compute. The results are given in tabular form in Table 1.

In passing we note that

$$\Delta_{2,4-6,5} = 0, \quad \Delta_{3,1,4+6} = 0 \quad \text{and} \quad \Delta_{6,7,8} = 0,$$

where Δ is a shorthand notation for determinant. Each of the combinations is a representation of the complete symmetry group of (4) [**1**] and it has been proposed that the determinant of a set of symmetries is zero if that set of symmetries is a representation of the complete symmetry group [**26**].

It is not surprising that the ratio integral corresponding to Γ_6, Γ_7 and Γ_8 is not to be found in the Table 1 since all of the elements in the matrices for which the determinant has been calculated are polynomial. The reciprocals of each of the nonzero determinants gives a multiplier and hence a route to the determination of a Lagrangian. One notes that the determinant which leads to the constant multiplier and so the standard representation for the Lagrangian of (4) is that of the vector field and the two solution symmetries. One further notes that there is scarcely a shortage of Lagrangians!

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
Γ_1	0	–	–	–	–			
Γ_2	1	0	–	–	–	–	–	–
Γ_3	0	I_1	0	–	–	–	–	–
Γ_4	$-I_1$	I_2	I_1^2	0	–	–	–	–
Γ_5	$-I_2$	0	I_1I_2	I_2^2	0	–	–	–
Γ_6	I_1	I_2	$-I_1^2$	$-2I_1I_2$	$-I_2^2$	0	–	–
Γ_7	$-I_1^2$	$-I_1I_2$	I_1^3	$-2I_1^2I_2$	$-I_1I_2^2$	0	0	–
Γ_8	$-I_1I_2$	$-I_2^2$	$I_1^2I_2$	$2I_1I_2^2$	$-I_2^3$	0	0	0

Table 1. The entries in the body of the table are the determinants of the two symmetries indicated with the vector field $\{1,\, p,\, 0\}$. The symmetries are once extended. To enable a compact representation the determinants are given in terms of the fundamental first integrals of system (12,13). They are $I_1 = p$ and $I_2 = pt - q$. This is possible since one of the determinants is constant and so all other determinants must be expressible in terms of first integrals.

5. DISCUSSION

The purpose of this paper has not been to berate those who devise new methods to solve problems. Nevertheless we find it quite disturbing that one of the motives for such devising has been an alleged failure of Noether's Theorem to provide the results desired. In this paper we have demonstrated by means of a very simple example how symmetries of the Euler-Lagrange equation which are not normally associated with Noether's Theorem are in fact Noether symmetries. The failure to recognize these symmetries as such is due to the assumption of an incorrect Lagrangian. We iterate that (3) is a relationship connecting the integral, the boundary term, the symmetry and the Lagrangian. When a Lagrangian is imposed, the only symmetries which can be used to construct an integral are those appropriate to that Lagrangian since it is from consideration of the invariance of the Action Integral for that specific Lagrangian that the corresponding symmetries are determined. The symmetries of the differential equation need not be Noether symmetries of any given Lagrangian. However, if one starts from the integral and some symmetry, (3) is a linear first-order partial differential equation for the required Lagrangian. As such its solution is guaranteed under quite slight constraints upon the coefficients in the equation. Consequently the existence of a first integral and a corresponding symmetry for a differential equation of appropriate order enables one to determine the Lagrangian for which the symmetry is a Noether symmetry and the integral a Noether integral.

As a final remark we note that our considerations have been restricted to point symmetries. In the context of the Jacobi Last Multiplier there is no reason for such a restriction apart from the greater ease of calculating point symmetries.

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