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THE SECOND ORDER ESTIMATE FOR THE SOLUTION TO A SINGULAR ELLIPTIC BOUNDARY VALUE PROBLEM

Ling Mi, Bin Liu

We study the second order estimate for the unique solution near the boundary to the singular Dirichlet problem $-\triangle u = b(x)g(u),\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $g \in C^1((0,\infty),(0,\infty))$, g is decreasing on $(0,\infty)$ with $\lim\limits_{s\to 0^+} g(s) = \infty$ and g is **normalized** regularly varying at zero with index $-\gamma$ ($\gamma > 1$), $b \in C^\alpha(\bar{\Omega})$ $(0 < \alpha < 1)$, is positive in Ω, may be vanishing on the boundary. Our analysis is based on Karamata regular variation theory.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the second order estimate for the unique solution near the boundary to the following singular boundary value problem

$$-\triangle u = b(x)g(u),\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, b satisfies

($\mathbf{b_1}$) $b \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$, and is positive in Ω,

($\mathbf{b_2}$) there exist $k \in \Lambda$ and $B_0 \in \mathbb{R}$ such that

$$b(x) = k^2(d(x))(1 + B_0 d(x) + o(d(x)))\ \text{ near } \partial\Omega,$$

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where $d(x) = \mathrm{dist}(x, \partial\Omega)$, Λ denotes the set of all positive non-decreasing functions in $C^1(0, \delta_0)$ which satisfy

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K(t)}{k(t)}\right) := C_k \in (0,1], \quad K(t) = \int_0^t k(s)\mathrm{d}s,$$

and g satisfies

($\mathbf{g_1}$) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$ and g is decreasing on $(0,\infty)$;

($\mathbf{g_2}$) there exist $\gamma > 1$ and a function $f \in C^1(0,a_1] \cap C[0,a_1]$ for $a_1 > 0$ small enough such that

$$\frac{-g'(s)s}{g(s)} := \gamma + f(s) \ \text{ with } \lim_{s\to 0^+} f(s) = 0, \ s \in (0,a_1],$$

i.e.,

$$g(s) = c_0 s^{-\gamma} \exp\left(\int_s^{a_1} \frac{f(\nu)}{\nu}\mathrm{d}\nu\right), \ s \in (0,a_1], \ c_0 > 0;$$

($\mathbf{g_3}$) there exists $\eta \geq 0$ such that

$$\lim_{s\to 0^+} \frac{f'(s)s}{f(s)} = \eta.$$

The problem (1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as the theory of heat conduction in electrical materials (see [**12**]-[**24**]) and has been discussed and extended by many authors in many contexts, for instance, the existence, uniqueness, regularity and boundary behavior of solutions, see, [**12**]-[**36**] and the references therein.

For $b \equiv 1$ in Ω and g satisfying (g_1), Fulks and Maybee [**12**], Stuart [**27**], Crandall, Rabinowitz and Tartar [**7**] derived that problem (1) has a unique solution $u \in C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$. Moreover, in [**7**], the following result was established: if $\phi_1 \in C[0,\delta_0] \cap C^2(0,\delta_0]$ is the local solution to the problem

$$-\phi_1''(t) = g(\phi_1(t)), \ \phi_1(t) > 0, \ 0 < t < \delta_0, \ \phi_1(0) = 0, \tag{2}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\phi_1(d(x)) \leq u(x) \leq c_2\phi_1(d(x)) \ \text{ near } \ \partial\Omega.$$

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u has the property

$$c_1(d(x))^{2/(1+\gamma)} \leq u(x) \leq c_2(d(x))^{2/(1+\gamma)} \ \text{ near } \ \partial\Omega. \tag{3}$$

By constructing global subsolutions and supersolutions, Lazer and McKenna [**20**] showed that (3) continued to hold on $\bar{\Omega}$. Then, $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. This is a basic characteristic of problem (1). Moreover, there is the following additional statement in [**20**].

($\mathbf{I_1}$) If, instead of $b \equiv 1$, we assume that $0 < \theta_1 \leq b(x)(\varphi_1(x))^{\varpi} \leq \theta_2$ for $x \in \Omega$, where θ_1 and θ_2 are positive constants, $\varpi \in (0, 2)$, and φ_1 is the first eigenfunction, corresponding to the first eigenvalue λ_1 of the Laplace operator with Dirichlet boundary conditions and $\gamma > 1$, then there exist positive constants θ_3 and θ_4 (θ_3 is small and θ_4 is large) such that

$$\theta_3(\varphi_1(x))^{\frac{2}{\gamma+1}} \leq u(x) \leq \theta_4(\varphi_1(x))^{\frac{2-\varpi}{\gamma+1}}, \ \forall x \in \Omega.$$

Giarrusso and Porru [**13**], Berhanu, Gladiali and Porru [**3**], Berhanu, Cuccu and Porru [**4**], McKenna and Reichel [**22**], Anedda [**1**], Anedda and Porru [**2**], Ghergu and Rădulescu [**14**] considered the first and second order expansions of the solution near the boundary. Specifically, when the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [**13**] proved that if g satisfies the following conditions

($\mathbf{g_1'}$) $\displaystyle\int_0^1 g(s)\mathrm{d}s = \infty, \ \int_1^\infty g(s)\mathrm{d}s < \infty, \ \ G_1(t) := \int_t^\infty g(s)\mathrm{d}s;$

($\mathbf{g_2'}$) there exist positive constants δ and $M > 1$ such that

$$G_1(t) < MG_1(2t), \ \forall t \in (0, \delta),$$

then for the unique solution u of problem (1)

$$|u(x) - \phi_2(d(x))| < c_0 d(x) \ \text{ near } \ \partial\Omega, \tag{4}$$

where c_0 is a suitable positive constant and $\phi_2 \in C[0, \infty) \cap C^2(0, \infty)$ is the unique solution of

$$\int_0^{\phi_2(t)} \frac{\mathrm{d}\nu}{\sqrt{2G_1(\nu)}} = t, \ t > 0. \tag{5}$$

Later, for $b \equiv 1$ on Ω, $g(u) = u^{-\gamma}$ with $\gamma > 0$, Berhanu, Gladiali and Porru [**3**] showed the following result for $\gamma > 1$

(i) $$\left| \frac{u(x)}{(d(x))^{2/(1+\gamma)}} - \left(\frac{(1+\gamma)^2}{2(\gamma-1)} \right)^{1/(1+\gamma)} \right| < c_3 (d(x))^{(\gamma-1)/(1+\gamma)} \ \text{ near } \ \partial\Omega.$$

Then, Berhanu, Cuccu and Porru [**4**] obtained the following results on a sufficiently small neighborhood of $\partial\Omega$;

(ii) for $\gamma = 1$,

$$u(x) = \phi_1(d(x)) \left(1 + A(x)(-\ln(d(x)))^{-\beta}\right) \ \text{ near } \ \partial\Omega,$$

where ϕ_1 is the solution of problem (2) with $\gamma = 1$, $\phi_1(t) \approx t\sqrt{-2\ln t}$ near $t = 0$, $\beta \in (0, 1/2)$ and A is bounded;

(iii) for $\gamma \in (1, 3)$,

$$u(x) = \left(\frac{(1+\gamma)^2}{2(\gamma-1)}\right)^{1/(1+\gamma)} (d(x))^{2/(1+\gamma)} \big(1 + A(x)(d(x))^{2(\gamma-1)/(1+\gamma)}\big) \text{ near } \partial\Omega;$$

(iv) for $\gamma = 3$,

$$u(x) = \sqrt{2d(x)}\,\big(1 - A(x)d(x)\ln(d(x))\big) \quad \text{near} \quad \partial\Omega.$$

For $\gamma > 3$, McKenna and Reichel [**22**] proved that

$$\left|\frac{u(x)}{(d(x))^{2/(1+\gamma)}} - \left(\frac{(1+\gamma)^2}{2(\gamma-1)}\right)^{1/(1+\gamma)}\right| < c_4(d(x))^{(\gamma+3)/(1+\gamma)} \quad \text{near} \quad \partial\Omega.$$

On the other hand, Cîrstea and Rădulescu [**9**]-[**11**] introducd a unified new appoach via the Karamata regular variation theory, to study the boundary behavior and uniqueness of solutions for boundary blow-up elliptic problems.

Let $\beta > 0$, we define

$$\Lambda_{1,\beta} = \left\{k \in \Lambda, \lim_{t\to 0^+} (-\ln t)^\beta \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K(t)}{k(t)}\right) - C_k\right) = D_{1k} \in \mathbb{R}\right\};$$

$$\Lambda_2 = \left\{k \in \Lambda, \lim_{t\to 0^+} t^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K(t)}{k(t)}\right) - C_k\right) = D_{2k} \in \mathbb{R}\right\}.$$

Recently, when g, b satisfy (g_1)-(g_3) and (b_1)-(b_2), using the Karamata regular variation theory, Zhang [**31**] proved that the two-term asymptotic expansion of the unique solution u near $\partial\Omega$ only depends on the distance function $d(x)$ and the above chosen subclasses for $k \in \Lambda$ under the following hypotheses:

($\mathbf{H_1}$) $\eta = 0$ in (g_3);

($\mathbf{H_2}$) there exist $\sigma \in \mathbb{R}$ such that

$$\lim_{s\to 0^+} (-\ln s)^\beta f(s) = \sigma,$$

where β is the parameter used in the definition of $\Lambda_{1,\beta}$;

($\mathbf{H_3}$) $C_k(\gamma + 1) > 2$.

However, Zhang [**31**] only considered the condition $\eta = 0$ in (g_3).

Inspired by the above works, in this paper we also consider the two-term asymptotic expansion of the unique solution u of problem (1) near $\partial\Omega$. We consider not only the condition $\eta = 0$ in (g_3) but also the condition $\eta > 0$ in (g_3). In [**31**], Zhang mainly used the solution to the problem

$$\int_0^{\psi(t)} \frac{\mathrm{d}s}{\sqrt{2G(s)}} = t,\ G(t) = \int_t^b g(s)\mathrm{d}s, \quad b > 0,\ t \in (0, b),$$

to estimate the boundary behavior of solutions to problem (1) while the key to our estimates in this paper is the solution to the problem

$$\int_0^{\phi(t)} \frac{\mathrm{d}s}{g(s)} = t,\ t > 0. \tag{6}$$

Our main results are summarized as follows.

Theorem 1. *Let g satisfy* (g_1)-(g_3), *b satisfy* (b_1)-(b_2) *and* (H_3) *holds. Suppose that $k \in \Lambda_{1,\beta}$ and $\eta > 0$ in (g_3), then for the unique solution u of problem* (1) *and all x in a neighborhood of $\partial\Omega$ it holds that*

$$u(x) = \xi_0 \phi(K^2(d(x))) \left(1 + A_0(-\ln(d(x)))^{-\beta} + o((-\ln(d(x)))^{-\beta})\right), \tag{7}$$

where ϕ is uniquely determined by (6) *and*

$$\xi_0 = \left(\frac{\gamma+1}{2C_k(\gamma+1)-4}\right)^{1/(1+\gamma)},\quad A_0 = -\frac{D_{1k}}{C_k(\gamma+1)-2}. \tag{8}$$

Theorem 2. *Let g satisfy* (g_1)-(g_3), *b satisfy* (b_1)-(b_2) *and* (H_1)-(H_3) *hold.*

- **(i)** *Suppose that $k \in \Lambda_{1,\beta}$, then for the unique solution u of problem* (1) *and all x in a neighborhood of $\partial\Omega$ it holds that*

$$u(x) = \xi_0 \phi(K^2(d(x))) \left(1 + A_1(-\ln(d(x)))^{-\beta} + o((-\ln(d(x)))^{-\beta})\right), \tag{9}$$

where ϕ is uniquely determined by (6), *ξ_0 is in* (8) *and*

$$A_1 = -\frac{2D_{1k} - A_2}{2C_k(\gamma+1)-4} \text{ with } A_2 = -A_3\left(4\sigma(\gamma+1)^{-2} + \sigma\xi_0^{-(\gamma+1)}\ln\xi_0\right),$$

$$A_3 = 2^{-\beta}(C_k(\gamma+1))^{\beta}.$$

- **(ii)** *Suppose that $k \in \Lambda_2$, then* **(i)** *still holds, where*

$$A_1 = \frac{A_2}{2C_k(\gamma+1)-4}.$$

Remark 1 (Existence, [**33**], Theorem 4.1). Let $b \in C^{\alpha}_{\ell oc}(\Omega)$ for some $\alpha \in (0,1)$, be nonnegative and nontrivial on Ω. If g satisfies (g_1), then problem (1) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ if and only if the linear problem $-\Delta w = b(x),\ w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0$ has a unique solution $w_0 \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$.

The outline of this paper is as follows. In section 2 we give some preparation. The proofs of Theorem 1-2 will be given in section 3.

2. PREPARATION

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in the theory of stochastic process (see [**23**], [**26**] and [**30**] and the references therein.). In this section, we give a brief account of the definition and properties of regularly varying functions involved in our paper (see [**23**], [**26**] and [**30**]).

Definition 1. *A positive measurable function g defined on $(0, a)$, for some $a > 0$, is called* **regularly varying at zero** *with index ρ, written as $g \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,*

$$\lim_{t \to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\rho. \tag{10}$$

In particular, when $\rho = 0$, g is called **slowly varying at zero**.

Clearly, if $g \in RVZ_\rho$, then $L(t) := g(t)/t^\rho$ is slowly varying at zero.

Some basic examples of slowly varying functions at zero are

(**i**) every measurable function on $(0, a)$ which has a positive limit at zero;

(**ii**) $(-\ln t)^p$ and $\big(\ln(-\ln t)\big)^p$, $p \in \mathbb{R}$;

(**iii**) $e^{(-\ln t)^p}$, $0 < p < 1$.

Proposition 1 (Uniform convergence theorem). *If $g \in RVZ_\rho$, then* (10) *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2 < a$.*

Proposition 2 (Representation theorem). *A function L is slowly varying at zero if and only if it can be written in the form*

$$L(t) = y(t) \exp\left(\int_t^{a_1} \frac{f(\nu)}{\nu} \mathrm{d}\nu\right), \; t \in (0, a_1), \tag{11}$$

for some $a_1 \in (0, a)$, where the functions f and y are measurable and for $t \to 0^+$, $f(t) \to 0$ and $y(t) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(t) = c_0 \exp\left(\int_t^{a_1} \frac{f(\nu)}{\nu} d\nu\right), \; t \in (0, a_1), \tag{12}$$

is **normalized** slowly varying at zero and

$$g(t) = c_0 t^\rho \hat{L}(t), \; t \in (0, a_1), \tag{13}$$

is **normalized** regularly varying at zero with index ρ (and written $g \in NRVZ_\rho$). A function $g \in RVZ_\rho$ belongs to $NRVZ_\rho$ if and only if

$$g \in C^1(0, a_1) \text{ for some } a_1 > 0 \text{ and } \lim_{t \to 0^+} \frac{t g'(t)}{g(t)} = \rho. \tag{14}$$

Proposition 3. *If functions L, L_1 are slowly varying at zero, then*

(i) L^ρ *(for every* $\rho \in \mathbb{R}$*),* $c_1 L + c_2 L_1$ $(c_1 \geq 0, c_2 \geq 0$ *with* $c_1 + c_2 > 0)$*,* $L \circ L_1$ *(if* $L_1(t) \to 0$ *as* $t \to 0^+$*), are also slowly varying at zero.*

(ii) *For every* $\rho > 0$ *and* $t \to 0^+$*,*

$$t^\rho L(t) \to 0, \quad t^{-\rho} L(t) \to \infty.$$

(iii) *For* $\rho \in \mathbb{R}$ *and* $t \to 0^+$*,* $\ln(L(t))/\ln t \to 0$ *and* $\ln(t^\rho L(t))/\ln t \to \rho$*.*

Proposition 4. *If* $g_1 \in RVZ_{\rho_1},\ g_2 \in RVZ_{\rho_2}$ *with* $\lim\limits_{t\to 0^+} g_2(t) = 0$*, then* $g_1 \circ g_2 \in RVZ_{\rho_1\rho_2}$*.*

Proposition 5 (Asymptotic behavior)**.** *If a function L is slowly varying at zero, then for* $a > 0$ *and* $t \to 0^+$*,*

(i) $\displaystyle\int_0^t s^\rho L(s)\mathrm{d}s \cong (\rho+1)^{-1}\ t^{1+\rho}\ L(t),\ \ for\ \rho > -1;$

(ii) $\displaystyle\int_t^a s^\rho L(s)\mathrm{d}s \cong (-\rho-1)^{-1}\ t^{1+\rho}\ L(t),\ \ for\ \rho < -1.$

Our results in this section are summarized as follows.

Lemma 1. *Let* $k \in \Lambda$*. Then*

(i) $\displaystyle\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0, \quad \lim_{t\to 0^+} \frac{tk(t)}{K(t)} = C_k^{-1},$ *i.e.,* $K \in NRVZ_{C_k^{-1}}$;

(ii) $\displaystyle\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \frac{1-C_k}{C_k},$ *i.e.,* $k \in NRVZ_{(1-C_k)/C_k}$; $\displaystyle\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k;$

(iii) $\displaystyle\lim_{t\to 0^+} (-\ln t)^\beta \left(\frac{K(t)k'(t)}{k^2(t)} - (1-C_k)\right) = -D_{1k},$ *if* $k \in \Lambda_{1,\beta}$;

(iv) $\displaystyle\lim_{t\to 0^+} t^{-1} \left(\frac{K(t)k'(t)}{k^2(t)} - (1-C_k)\right) = -D_{2k},$ *if* $k \in \Lambda_2$.

Proof. The proof is similar to the proof of Lemma 2.1 in [**31**], so we omit it.

Lemma 2. *If g satisfies* (g_1)-(g_3)*, then*

(i)
$$\int_0^a \frac{\mathrm{d}s}{g(s)} < \infty, \quad for\ some\ a > 0;$$

(ii)
$$\lim_{t\to 0^+} g'(t) \int_0^t \frac{\mathrm{d}s}{g(s)} = -\frac{\gamma}{\gamma+1}\ \ and\ \lim_{t\to 0^+} \frac{g(t)\int_0^t \frac{\mathrm{d}s}{g(s)}}{t} = \frac{1}{\gamma+1}.$$

Proof. (i) (g_2) implies that $g \in NRVZ_{-\gamma}$ with $\gamma > 1$, so $g(s) = c_0 s^{-\gamma}\hat{L}(s),\ s \in (0, a_1)$, where $\hat{L}$ is normalized slowly varying at zero and $c_0 > 0$. (i) is obvious due to Propositions 5(i) and 3(ii).
(ii) Also

$$g'(t)\int_0^t \frac{\mathrm{d}s}{g(s)} \sim \frac{tg'(t)}{g(t)}\frac{1}{\gamma+1} = -\frac{\gamma}{\gamma+1}$$

and

$$\frac{g(t)}{t}\int_0^t \frac{\mathrm{d}s}{g(s)} \sim \frac{t^{-\gamma}}{L(t)}\frac{t^{\gamma+1}L(t)}{t(\gamma+1)} = \frac{1}{\gamma+1}.$$

Lemma 3. *Let g satisfy* (g_1)-(g_3). *If $\eta = 0$ in* (g_3), *suppose that* (H_2) *holds. Then*

(i) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\Big(\frac{tg'(t)}{g(t)} + \gamma\Big) = -\sigma I_{\eta>0},$$

(ii) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{\int_0^t \frac{ds}{g(s)}}{\frac{t}{g(t)}} - \frac{1}{\gamma+1}\right) = -\frac{\sigma}{(\gamma+1)^2}I_{\eta>0},$$

(iii) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(g'(t)\int_0^t \frac{\mathrm{d}s}{g(s)} + \frac{\gamma}{\gamma+1}\right) = -\frac{\sigma}{(\gamma+1)^2}I_{\eta>0},$$

(iv) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{g(\xi_0 t)}{\xi_0 g(t)} - \xi_0^{-(\gamma+1)}\right) = -\sigma\xi_0^{-(\gamma+1)}\ln\xi_0 I_{\eta>0},$$

Proof. When $f \in NRVZ_{\eta}$ with $\eta > 0$, by Proposition 3 (ii), $\lim_{t\to 0^+}(-\ln t)^{\beta}f(t) = 0$, and when $\eta = 0$, by hypothesis (H_2), $\lim_{t\to 0^+}(-\ln t)^{\beta}f(t) = \sigma$.
(i) By $\frac{tg'(t)}{g(t)} + \gamma = -f(t)$, we see that (i) holds.
(ii) By (g_2) and a simple calculation, we obtain

$$s\left(\frac{1}{g(s)}\right)' = \frac{\gamma}{g(s)} + \frac{f(s)}{g(s)},\ s \in (0, a_1]. \tag{15}$$

Since $g \in NRVZ_{-\gamma}$ with $\gamma > 1$, by Proposition 3 (ii), we have $\lim_{t\to 0^+}\frac{t}{g(t)} = 0$. Integrating (15) from 0 to t and integrating by parts, we get

$$\frac{t}{g(t)} = (\gamma+1)\int_0^t \frac{\mathrm{d}s}{g(s)} + \int_0^t \frac{f(s)}{g(s)}\mathrm{d}s, \quad t \in (0, a_1],$$

i.e.,

$$\frac{\int_0^t \frac{\mathrm{d}s}{g(s)}}{\frac{t}{g(t)}} - \frac{1}{\gamma+1} = -\frac{f(t)}{\gamma+1}\frac{\int_0^t \frac{f(s)}{g(s)}\mathrm{d}s}{t\frac{f(t)}{g(t)}},\ t \in (0, a_1].$$

Since $g \in NRVZ_{-\gamma}$, $f \in NRVZ_{\eta}$, we obtain by Proposition 5 that

$$\lim_{t\to 0^+} \frac{\int_0^t \frac{f(s)}{g(s)}\mathrm{d}s}{t\frac{f(t)}{g(t)}} = \frac{1}{\gamma+\eta+1}.$$

Thus,

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{\int_0^t \frac{\mathrm{d}s}{g(s)}}{\frac{t}{g(t)}} - \frac{1}{\gamma+1}\right) = -\frac{1}{\gamma+1}\lim_{t\to 0^+}(-\ln t)^{\beta} f(t) \lim_{t\to 0^+}\frac{\int_0^t \frac{f(s)}{g(s)}\mathrm{d}s}{t\frac{f(t)}{g(t)}} = \sigma_2.$$

(iii) By a simple calculation, we have

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(g'(t)\int_0^t \frac{\mathrm{d}s}{g(s)} + \frac{\gamma}{\gamma+1}\right) = \lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{tg'(t)}{g(t)}\frac{\int_0^t \frac{\mathrm{d}s}{g(s)}}{\frac{t}{g(t)}} + \frac{\gamma}{\gamma+1}\right)$$

$$= \lim_{t\to 0^+}(-\ln t)^{\beta}\left(\Big(\frac{tg'(t)}{g(t)}+\gamma\Big)\left(\frac{\int_0^t \frac{\mathrm{d}s}{g(s)}}{\frac{t}{g(t)}} - \frac{1}{\gamma+1}\right)\right.$$

$$\left.+\ \frac{1}{\gamma+1}\Big(\frac{tg'(t)}{g(t)}+\gamma\Big) - \gamma\left(\frac{\int_0^t \frac{\mathrm{d}s}{g(s)}}{\frac{t}{g(t)}} - \frac{1}{\gamma+1}\right)\right).$$

Hence, by (i)-(ii), we get

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(g'(t)\int_0^t \frac{\mathrm{d}s}{g(s)} + \frac{\gamma}{\gamma+1}\right) = \sigma_3.$$

(iv) When $\xi_0 = 1$, the result is obvious. Now suppose that $\xi_0 \neq 1$. By (g_2), we obtain

$$\frac{g(\xi_0 t)}{\xi_0 g(t)} - \xi_0^{-(\gamma+1)} = \xi_0^{-(\gamma+1)}\left(\exp\Big(\int_{\xi_0 t}^t \frac{f(\nu)}{\nu}\mathrm{d}\nu\Big) - 1\right).$$

Note that

$$\lim_{t\to 0^+}\frac{f(ts)}{s} = 0 \ \text{ and } \ \lim_{t\to 0^+}\frac{f(ts)}{f(t)s} = s^{\eta-1}$$

uniformly with respect to $s \in [1,\xi_0]$ or $s \in [\xi_0, 1]$.
So,

$$\lim_{t\to 0^+}\int_{\xi_0 t}^t \frac{f(\nu)}{\nu}\mathrm{d}\nu = \lim_{t\to 0^+}\int_{\xi_0}^1 \frac{f(ts)}{s}\mathrm{d}s = 0$$

and

$$\lim_{t\to 0^+}\int_{\xi_0}^{1}\frac{f(ts)}{f(t)s}\mathrm{d}s=\int_{\xi_0}^{1}s^{\eta-1}\mathrm{d}s=\chi,$$

where

$$\chi=\begin{cases}-\ln\xi_0, & \text{if } \ \eta=0;\\ \dfrac{1}{\eta}(1-\xi_0^{\eta}), & \text{if } \ \eta>0.\end{cases}$$

Since $e^r-1\sim r$ as $r\to 0$, it follows that

$$\frac{g(\xi_0 t)}{\xi_0 g(t)}-\xi_0^{-(\gamma+1)}\sim\xi_0^{-(\gamma+1)}\int_{\xi_0 t}^{t}\frac{f(\nu)}{\nu}\mathrm{d}\nu \ \text{ as } \ t\to 0.$$

Hence,

$$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{g(\xi_0 t)}{\xi_0 g(t)}-\xi_0^{-(\gamma+1)}\right)=\xi_0^{-(\gamma+1)}\lim_{t\to 0^+}(-\ln t)^{\beta}f(t)\lim_{t\to 0^+}\int_{\xi_0}^{1}\frac{f(ts)}{f(t)s}\mathrm{d}s=\sigma_4.$$

Lemma 4. *Suppose that g satisfies* (g_1)-(g_3) *and let ϕ be the solution to the problem*

$$\int_0^{\phi(t)}\frac{\mathrm{d}s}{g(s)}=t,\ \forall\ t>0.$$

Then

(i) $\phi'(t)=g(\phi(t))$, $\phi(t)>0$, $t>0$, $\phi(0):=\lim_{t\to 0^+}\phi(t)=0$, *and* $\phi''(t)=g(\phi(t))g'(\phi(t))$, $t>0$;

(ii) $\phi\in NRVZ_{\frac{1}{\gamma+1}}$;

(iii) $\phi'=g\circ\phi\in NRVZ_{-\frac{\gamma}{\gamma+1}}$;

(iv) $\lim_{t\to 0^+}\dfrac{\ln t}{\ln(\phi(K^2(t)))}=\dfrac{C_k(\gamma+1)}{2}$, *if* $k\in\Lambda$,

(v) $\lim_{t\to 0^+}(-\ln t)^{\beta}\dfrac{t}{\phi(K^2(t))}=0$, *if* $k\in\Lambda$ *and* $C_k(\gamma+1)>2$.

Proof. By the definition of ϕ and a direct calculation, we can prove (i).
Let $u=\phi(t)$, by Lemma 2, we have that

$$\lim_{t\to 0^+}\frac{t\phi''(t)}{\phi'(t)}=\lim_{t\to 0^+}tg'(\phi(t))=\lim_{u\to 0^+}g'(u)\int_0^u\frac{\mathrm{d}s}{g(s)}=-\frac{\gamma}{\gamma+1},$$

and

$$\lim_{t\to 0^+}\frac{t\phi'(t)}{\phi(t)}=\lim_{t\to 0^+}\frac{tg(\phi(t))}{\phi(t)}=\lim_{u\to 0^+}\frac{g(u)}{u}\int_0^u\frac{\mathrm{d}s}{g(s)}=\frac{1}{\gamma+1},$$

i.e., $\phi' = g\circ\phi \in NRVZ_{-\frac{\gamma}{\gamma+1}}$ and $\phi \in NRVZ_{\frac{1}{\gamma+1}}$ and (iii) follows.
Since $K \in NRVZ_{C_k^{-1}}$ and $\phi \in NRVZ_{\frac{1}{\gamma+1}}$, we see by Proposition 3 (iii) that (iv) holds.

By (iv) and Proposition 4, we have that $\phi\circ K^2 \in NRVZ_{\frac{2}{C_k(\gamma+1)}}$ and $\frac{t}{\phi(K^2(t))} \in NRVZ_{\frac{C_k(\gamma+1)-2}{C_k(\gamma+1)}}$. Since $C_k(\gamma+1) > 2$, (v) follows by Proposition 3 (ii).

Lemma 5. *Suppose that g satisfies* (g_1)-(g_3), *b satisfies* (b_1)-(b_2) *and* (H_3) *holds. If $k \in \Lambda_{1,\beta}$, $\eta > 0$ in* (g_3) *and ϕ is the solution to the problem*

$$\int_0^{\phi(t)} \frac{\mathrm{d}s}{g(s)} = t,\ \forall\ t > 0,$$

then

(i) $$\lim_{t\to 0^+}(-\ln t)^\beta\left(\frac{K^2(t)\phi''(K^2(t))}{\phi'(K^2(t))} + \frac{\gamma}{\gamma+1}\right) = 0;$$

(ii) $$\lim_{t\to 0^+}(-\ln t)^\beta\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))} - \xi_0^{-(\gamma+1)}\right) = 0.$$

Proof. (i) By the definition of ϕ, Lemma 3 (iii) and Lemma 4 (iv), we arrive at

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^\beta\left(\frac{K^2(t)\phi''(K^2(t))}{\phi'(K^2(t))} + \frac{\gamma}{\gamma+1}\right)\\
&= \lim_{t\to 0^+}(-\ln t)^\beta\left(g'(\phi(K^2(t)))\int_0^{\phi(K^2(t))}\frac{\mathrm{d}s}{g(s)} + \frac{\gamma}{\gamma+1}\right)\\
&= \lim_{t\to 0^+}(-\ln(\phi(K^2(t))))^\beta\left(g'(\phi(K^2(t)))\int_0^{\phi(K^2(t))}\frac{\mathrm{d}s}{g(s)} + \frac{\gamma}{\gamma+1}\right)\\
&\times \lim_{t\to 0^+}\left(\frac{\ln t}{\ln\phi(K^2(t))}\right)^\beta = 0.
\end{aligned}$$

(ii) By Lemma 3 (iv) and Lemma 4 (iv), we infer that

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^\beta\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))} - \xi_0^{-(\gamma+1)}\right)\\
&= \lim_{t\to 0^+}(-\ln(\phi(K^2(t))))^\beta\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))} - \xi_0^{-(\gamma+1)}\right)\lim_{t\to 0^+}\left(\frac{\ln t}{\ln\phi(K^2(t))}\right)^\beta = 0.
\end{aligned}$$

Lemma 6. *Suppose that g satisfies* (g_1)-(g_3), *b satisfies* (b_1)-(b_2) *and* (H_1)-(H_3) *hold. If ϕ is the solution to the problem*

$$\int_0^{\phi(t)} \frac{\mathrm{d}s}{g(s)} = t,\ \forall\ t > 0,$$

then

(i) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{K^2(t)\phi''(K^2(t))}{\phi'(K^2(t))}+\frac{\gamma}{\gamma+1}\right)=-\frac{A_3\sigma}{(\gamma+1)^2};$$

(ii) $$\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))}-\xi_0^{-(\gamma+1)}\right)=-A_3\sigma\xi_0^{-(\gamma+1)}\ln\xi_0,$$

where $A_3=2^{-\beta}(C_k(1+\gamma))^{\beta}$.

Proof. (i) By the definition of ϕ, Lemma 3 (iii) and Lemma 4 (iv), we find that

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{K^2(t)\phi''(K^2(t))}{\phi'(K^2(t))}+\frac{\gamma}{\gamma+1}\right)\\
&=\lim_{t\to 0^+}(-\ln t)^{\beta}\left(g'(\phi(K^2(t)))\int_0^{\phi(K^2(t))}\frac{\mathrm{d}s}{g(s)}+\frac{\gamma}{\gamma+1}\right)\\
&=\lim_{t\to 0^+}(-\ln\phi(K^2(t)))^{\beta}\left(g'(\phi(K^2(t)))\int_0^{\phi(K^2(t))}\frac{\mathrm{d}s}{g(s)}+\frac{\gamma}{\gamma+1}\right)\\
&\quad\lim_{t\to 0^+}\left(\frac{\ln t}{\ln\phi(K^2(t))}\right)^{\beta}=-\frac{A_3\sigma}{(\gamma+1)^2}.
\end{aligned}$$

(ii) By Lemma 3 (iv) and Lemma 4 (iv), we obtain

$$\begin{aligned}
&\lim_{t\to 0^+}(-\ln t)^{\beta}\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))}-\xi_0^{-(\gamma+1)}\right)\\
&=\lim_{t\to 0^+}(-\ln\phi(K^2(t)))^{\beta}\left(\frac{g(\xi_0\phi(K^2(t)))}{\xi_0 g(\phi(K^2(t)))}-\xi_0^{-(\gamma+1)}\right)\\
&\quad\lim_{t\to 0^+}\left(\frac{\ln t}{\ln\phi(K^2(t))}\right)^{\beta}=-A_3\sigma\xi_0^{-(\gamma+1)}\ln\xi_0
\end{aligned}$$

3. PROOFS OF THEOREMS

In this section, we prove Theorems 1-2.

First we need the following result.

Lemma 7 (the comparison principle, [**19**], Theorems 10.1 and 10.2)**.** *Let* $\Psi(x,s,\xi)$ *satisfy the following two conditions*

($\mathbf{D_1}$) Ψ *is non-increasing in* s *for each* $(x,\xi)\in\Omega\times\mathbb{R}^N$;

($\mathbf{D_2}$) Ψ *is continuously differentiable with respect to the* ξ *variables in* $\Omega\times(0,\infty)\times\mathbb{R}^N$.

If $u,v\in C(\bar{\Omega})\cap C^2(\Omega)$ *satisfies* $\Delta u+\Psi(x,u,\nabla u)\geq\Delta v+\Psi(x,v,\nabla v)$ *in* Ω *and* $u\leq v$ *on* $\partial\Omega$*, then* $u\leq v$ *in* Ω.

3.1. Proof of Theorem 1

Fix $\varepsilon > 0$. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is C^2-smooth, choose $\delta_1 \in (0, \delta_0)$ such that $d \in C^2(\Omega_{\delta_1})$ and

$$|\nabla d(x)| = 1,\ \Delta d(x) = -(N-1)H(\bar{x}) + o(1),\ \forall\, x \in \Omega_{\delta_1}. \tag{16}$$

where, for $x \in \Omega_{\delta_1}$, $\bar{x}$ denotes the unique point of the boundary such that $d(x) = |x - \bar{x}|$ and $H(\bar{x})$ denotes the mean curvature of the boundary at that point.

Let

$$w_\pm = \xi_0 \phi(K^2(d(x)))\left(1 + (A_0 \pm \varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x \in \Omega_{\delta_1}.$$

By the Lagrange mean value theorem, we obtain that there exist $\lambda_\pm \in (0,1)$ and

$$\Phi_\pm(d(x)) = \xi_0 \phi(K^2(d(x)))\left(1 + \lambda_\pm (A_0 \pm \varepsilon)(-\ln(d(x)))^{-\beta}\right)$$

such that for $x \in \Omega_{\delta_1}$

$$g(w_\pm(x)) = g(\xi_0\phi(K^2(d(x)))) + \xi_0(A_0 \pm \varepsilon)\phi(K^2(d(x)))g'(\Phi_\pm(d(x)))(-\ln(d(x)))^{-\beta}.$$

Since $g \in NRVZ_{-\gamma}$, by Proposition 1 we obtain

$$\lim_{d(x)\to 0} \frac{g(\xi_0\phi(K^2(d(x))))}{g(\Phi_\pm(d(x)))} = \lim_{d(x)\to 0} \frac{g'(\xi_0\phi(K^2(d(x))))}{g'(\Phi_\pm(d(x)))} = 1.$$

Define $r = d(x)$ and

$$I_1(r) = (-\ln r)^\beta \left(4\frac{K^2(r)\phi''(K^2(r))}{\phi'(K^2(r))} + 2\frac{K(r)k'(r)}{k^2(r)} + \frac{g(\xi_0\phi(K^2(r)))}{\xi_0 g(\phi(K^2(r)))} + 2\right);$$

$$I_{2\pm}(r) = (A_0 \pm \varepsilon)\left(4\frac{K^2(r)\phi''(K^2(r))}{\phi'(K^2(r))} + 2\frac{K(r)k'(r)}{k^2(r)} + \frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^2(r)))}\right.$$
$$\left.\times \frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{\phi'(K^2(r))} + 2\right);$$

$$I_{3\pm}(x) = \beta(A_0 \pm \varepsilon)\frac{\phi(K^2(r))}{\phi'(K^2(r))k^2(r)}\big((\beta+1)(-\ln r)^{-2}r^{-2} + (-\ln r)^{-1}r^{-1}\Delta d(x)$$
$$-(-\ln r)^{-1}r^{-2}\big) + (B_0 \pm \varepsilon)(-\ln r)^\beta r \frac{g(\xi_0\phi(K^2(r)))}{\xi_0 g(\phi(K^2(r)))};$$

$$I_{4\pm}(x) = 2\frac{K(r)}{k(r)}\left((A_0 \pm \varepsilon)\big(\Delta d(x) + 2\beta(-\ln r)^{-1}r^{-1}\big) + \Delta d(x)(-\ln r)^\beta\right)$$
$$+(A_0 \pm \varepsilon)(B_0 \pm \varepsilon) r \frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^2(r)))}\frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{\phi'(K^2(r))}.$$

By (10), (14), Lemmas 1, 4 and 5, combining with the choices of $\xi_0,\ A_0$ in Theorem 1, we obtain the following lemma.

Lemma 8. *Suppose that g satisfies* (g_1)-(g_3), *b satisfies* (b_1)-(b_2) *and* (H_3) *holds. If $k \in \Lambda_{1,\beta}$ and $\eta > 0$ in* (g_3), *then*

(i) $\lim_{r\to 0} I_1(r) = -2D_{1k}$;

(ii) $\lim_{r\to 0} I_{2\pm}(r) = (A_0 \pm \varepsilon)(4 - 2C_k(\gamma+1))$;

(iii) $\lim_{d(x)\to 0} I_{3\pm}(x) = 0$;

(iv) $\lim_{d(x)\to 0} I_{4\pm}(x) = 0$;

(v) $\lim_{d(x)\to 0} (I_1(r) + I_{2\pm}(r) + I_{3\pm}(x) + I_{4\pm}(x)) = \pm\varepsilon(4 - 2C_k(\gamma+1))$.

Proof of Theorem 1. Let $v \in C^{2+\alpha}(\Omega) \cap C^1(\bar{\Omega})$ be the unique solution of the problem

$$-\Delta v = 1,\ v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0. \tag{17}$$

By the Hopf maximum principle [**19**], we see that

$$\nabla v(x) \neq 0,\ \forall x \in \partial\Omega \text{ and } c_5 d(x) \leq v(x) \leq c_6 d(x),\ \forall x \in \Omega, \tag{18}$$

where c_5, c_6 are positive constants.

By (b_1), (b_2), Lemma 1 and $K \in C[0,\delta_0)$ with $K(0) = 0$, we see that there exist $\delta_{1\varepsilon}, \delta_{2\varepsilon} \in \big(0, \min\{1, \delta_1\}\big)$ (which is corresponding to ε) sufficiently small such that

(I) $0 \leq K^2(r) \leq \delta_{1\varepsilon},\ r \in (0, \delta_{2\varepsilon})$;

(II) $k^2(d(x))(1 + (B_0 - \varepsilon)d(x)) \leq b(x) \leq k^2(d(x))(1 + (B_0 + \varepsilon)d(x)),\ x \in \Omega_{\delta_{1\varepsilon}}$;

(III) $I_1(r) + I_{2+}(r) + I_{3+}(x) + I_{4+}(x) \leq 0,\ \forall\ (x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0, \delta_{2\varepsilon})$;

(IV) $I_1(r) + I_{2-}(r) + I_{3-}(x) + I_{4-}(x) \geq 0,\ \forall\ (x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0, \delta_{2\varepsilon})$.

Now we define

$$\bar{u}_\varepsilon = \xi_0 \phi(K^2(d(x))) \left(1 + (A_0 + \varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x \in \Omega_{\delta_{1\varepsilon}}.$$

Then for $x \in \Omega_{\delta_{1\varepsilon}}$

$$g(\bar{u}_\varepsilon(x)) = g(\xi_0\phi(K^2(d(x)))) + \xi_0(A_0+\varepsilon)\phi(K^2(d(x)))g'(\Phi_+(d(x)))(-\ln(d(x)))^{-\beta},$$

where $\lambda_+ \in (0,1)$ and

$$\Phi_+(d(x)) = \xi_0\phi(K^2(d(x))) \left(1 + \lambda_+(A_0 + \varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x \in \Omega_{\delta_{1\varepsilon}}.$$

By Lemma 8 and a direct calculation, we see that for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\Delta\bar{u}_\varepsilon(x) + k^2(d(x))(1 + (B_0 + \varepsilon)d(x))g(\bar{u}_\varepsilon(x))$$

$$
\begin{aligned}
&= 4\xi_0\phi''(K^2(d(x)))K^2(d(x))k^2(d(x))\left(1+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right)\\
&\quad + 2\xi_0\phi'(K^2(d(x)))k^2(d(x))\left(1+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right)\\
&\quad + 2\xi_0\phi'(K^2(d(x)))K(d(x))k'(d(x))\left(1+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right)\\
&\quad + 2\xi_0\phi'(K^2(d(x)))K(d(x))k(d(x))\Delta d(x)\left(1+(A_0+\varepsilon)(-\ln(d(x)))^{-\beta}\right)\\
&\quad + 4\xi_0\beta(A_0+\varepsilon)\phi'(K^2(d(x)))K(d(x))k(d(x))(-\ln(d(x)))^{-\beta-1}(d(x))^{-1}\\
&\quad + \xi_0\beta(A_0+\varepsilon)\phi(K^2(d(x)))\left((\beta+1)(-\ln(d(x)))^{-\beta-2}(d(x))^{-2}\right.\\
&\quad \left.+(-\ln(d(x)))^{-\beta-1}(d(x))^{-1}\Delta d(x)-(-\ln(d(x)))^{-\beta-1}(d(x))^{-2}\right)\\
&\quad + k^2(d(x))\big(1+(B_0+\varepsilon)d(x)\big)\Big(g(\xi_0\phi(K^2(d(x))))\\
&\quad + \xi_0(A_0+\varepsilon)\phi(K^2(d(x)))g'(\Phi_+(d(x)))(-\ln(d(x)))^{-\beta}\Big)\\
&= \xi_0\phi'(K^2(d(x)))k^2(d(x))(-\ln(d(x)))^{-\beta}\big(I_1(r)+I_{2+}(r)+I_{3+}(x)+I_{4+}(x)\big)\le 0,
\end{aligned}
$$

where $r=d(x)$, i.e., $\bar u_\varepsilon$ is a supersolution of equation (1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we show that

$$
\underline{u}_\varepsilon=\xi_0\phi(K^2(d(x)))\left(1+(A_0-\varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x\in\Omega_{\delta_{1\varepsilon}},
$$

is a subsolution of equation (1) in $\Omega_{\delta_{1\varepsilon}}$.

Let $u\in C(\bar\Omega)\cap C^{2+\alpha}(\Omega)$ be the unique solution to problem (1). We assert that there exists M large enough such that

$$
u(x)\le Mv(x)+\bar u_\varepsilon(x),\quad \underline{u}_\varepsilon(x)\le u(x)+Mv(x),\quad x\in\Omega_{\delta_{1\varepsilon}}, \tag{19}
$$

where v is the solution of problem (17).

In fact, we can choose M large enough such that

$$
u(x)\le\bar u_\varepsilon(x)+Mv(x)\ \text{ and }\ \underline{u}_\varepsilon(x)\le u(x)+Mv(x)\ \text{ on }\ \{x\in\Omega: d(x)=\delta_{1\varepsilon}\}.
$$

We see by (g_1) that $\bar u_\varepsilon(x)+Mv(x)$ and $u(x)+Mv(x)$ are also supersolutions of equation (1) in $\Omega_{\delta_{1\varepsilon}}$. Since $u=\bar u_\varepsilon+Mv=u+Mv=\underline{u}_\varepsilon=0$ on $\partial\Omega$, (19) follows by (g_1) and Lemma 7. Hence, for $x\in\Omega_{\delta_{1\varepsilon}}$

$$
A_0-\varepsilon-\frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))}\le(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))}-1\right)
$$

and

$$
(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))}-1\right)\le A_0+\varepsilon+\frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))}.
$$

Consequently, by (18) and Lemma 4 (v),

$$
A_0-\varepsilon\le\liminf_{d(x)\to 0}(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))}-1\right);
$$

$$\limsup_{d(x)\to 0}(-\ln(d(x)))^{\beta}\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))}-1\right)\le A_0+\varepsilon.$$

Thus, letting $\varepsilon\to 0$, we obtain (7).

3.2. Proof of Theorem 2

As before, fix $\varepsilon>0$. For any $\delta>0$, we define $\Omega_\delta=\{x\in\Omega:0<d(x)<\delta\}$. Since Ω is C^2-smooth, choose $\delta_1\in(0,\delta_0)$ such that $d\in C^2(\Omega_{\delta_1})$ and (16) holds.

Let

$$w_\pm=\xi_0\phi(K^2(d(x)))\left(1+(A_1\pm\varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x\in\Omega_{\delta_1}.$$

By the Lagrange mean value theorem, we obtain that there exist $\lambda_\pm\in(0,1)$ and

$$\Phi_\pm(d(x))=\xi_0\phi(K^2(d(x)))\left(1+\lambda_\pm(A_1\pm\varepsilon)(-\ln(d(x)))^{-\beta}\right)$$

such that for $x\in\Omega_{\delta_1}$

$$g(w_\pm(x))=g(\xi_0\phi(K^2(d(x))))+\xi_0(A_1\pm\varepsilon)\phi(K^2(d(x)))g'(\Phi_\pm(d(x)))(-\ln(d(x)))^{-\beta}.$$

Since $g\in NRVZ_{-\gamma}$, by Proposition 1 we obtain

$$\lim_{d(x)\to 0}\frac{g(\xi_0\phi(K^2(d(x))))}{g(\Phi_\pm(d(x)))}=\lim_{d(x)\to 0}\frac{g'(\xi_0\phi(K^2(d(x))))}{g'(\Phi_\pm(d(x)))}=1.$$

Define $r=d(x)$ and

$$I_1(r)=(-\ln r)^{\beta}\left(4\frac{K^2(r)\phi''(K^2(r))}{\phi'(K^2(r))}+2\frac{K(r)k'(r)}{k^2(r)}+\frac{g(\xi_0\phi(K^2(r)))}{\xi_0 g(\phi(K^2(r)))}+2\right);$$

$$I_{2\pm}(r)=(A_1\pm\varepsilon)\left(4\frac{K^2(r)\phi''(K^2(r))}{\phi'(K^2(r))}+2\frac{K(r)k'(r)}{k^2(r)}+\frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^2(r)))}\right.$$
$$\left.\times\frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{\phi'(K^2(r))}+2\right);$$

$$I_{3\pm}(x)=\beta(A_1\pm\varepsilon)\frac{\phi(K^2(r))}{\phi'(K^2(r))k^2(r)}\big((\beta+1)(-\ln r)^{-2}r^{-2}+(-\ln r)^{-1}r^{-1}\Delta d(x)$$
$$-(-\ln r)^{-1}r^{-2}\big)+(B_0\pm\varepsilon)(-\ln r)^{\beta}r\frac{g(\xi_0\phi(K^2(r)))}{\xi_0 g(\phi(K^2(r)))};$$

$$I_{4\pm}(x)=2\frac{K(r)}{k(r)}\left((A_1\pm\varepsilon)\big(\Delta d(x)+2\beta(-\ln r)^{-1}r^{-1}\big)+\Delta d(x)(-\ln r)^{\beta}\right)$$
$$+(A_1\pm\varepsilon)(B_0\pm\varepsilon)r\frac{g'(\Phi_\pm(r))}{g'(\xi_0\phi(K^2(r)))}\frac{\phi(K^2(r))g'(\xi_0\phi(K^2(r)))}{\phi'(K^2(r))}.$$

By (10), (14), Lemmas 1, 4 and 6, combining with the choices of ξ_0, A_1, A_2, A_3 in Theorem 2, we obtain the following lemma.

Lemma 9. *Suppose that g satisfies* (g_1)-(g_3), *b satisfies* (b_1)-(b_2) *and* (H_1)-(H_3) *hold, then*

(i) $\lim_{r\to 0} I_1(r) = -2D_{1k} + A_2,$ *if* $k \in \Lambda_{1,\beta}$,

(ii) $\lim_{r\to 0} I_1(r) = A_2,$ *if* $k \in \Lambda_2$,

(iii) $\lim_{r\to 0} I_{2\pm}(r) = (A_1 \pm \varepsilon)(4 - 2C_k(\gamma+1))$;

(iv) $\lim_{d(x)\to 0} I_{3\pm}(x) = 0$;

(v) $\lim_{d(x)\to 0} I_{4\pm}(x) = 0$;

(vi) $\lim_{d(x)\to 0} \big(I_1(r) + I_{2\pm}(r) + I_{3\pm}(x) + I_{4\pm}(x)\big) = \pm\varepsilon(4 - 2C_k(\gamma+1))$.

Proof of Theorem 2. As in the proof of Theorem 1, suppose that

$$\bar{u}_\varepsilon = \xi_0 \phi(K^2(d(x)))\left(1 + (A_1+\varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x \in \Omega_{\delta_{1\varepsilon}}.$$

Then, by Lemma 9 and a direct calculation, we have for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\begin{aligned}&\Delta\bar{u}_\varepsilon(x) + k^2(d(x))\big(1 + (B_0+\varepsilon)d(x)\big)g(\bar{u}_\varepsilon(x))\\ &= \xi_0\phi'(K^2(d(x)))k^2(d(x))(-\ln(d(x)))^{-\beta}\big(I_1(r) + I_{2+}(r) + I_{3+}(x) + I_{4+}(x)\big) \le 0,\end{aligned}$$

where $r = d(x)$, i.e., $\bar{u}_\varepsilon$ is a supersolution of equation (1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we can show that

$$\underline{u}_\varepsilon = \xi_0 \phi(K^2(d(x)))\left(1 + (A_1-\varepsilon)(-\ln(d(x)))^{-\beta}\right),\ x \in \Omega_{\delta_{1\varepsilon}},$$

is a subsolution of equation (1) in $\Omega_{\delta_{1\varepsilon}}$.

As in the proof of Theorem 1, we obtain for $x \in \Omega_{\delta_{1\varepsilon}}$

$$A_1 - \varepsilon - \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))} \le (-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\right)$$

and

$$(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\right) \le A_1 + \varepsilon + \frac{Mv(x)(-\ln(d(x)))^\beta}{\xi_0\phi(K^2(d(x)))}.$$

Consequently, by (18) and Lemma 4 (v),

$$A_1 - \varepsilon \le \liminf_{d(x)\to 0}(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\right),$$

$$\limsup_{d(x)\to 0}(-\ln(d(x)))^\beta\left(\frac{u(x)}{\xi_0\phi(K^2(d(x)))} - 1\right) \le A_1 + \varepsilon.$$

Thus, letting $\varepsilon \to 0$, we obtain (9).

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School of Science, Linyi University,
Linyi, Shandong
P.R. China
and
School of Mathematical Science,
Peking University,
Beijing
P.R. China
E-mail: mi-ling@163.com

School of Mathematical Science,
Peking University,
Beijing
P.R. China
E-mail: bliu@math.pku.edu.cn

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