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ON THE DISTRIBUTION OF A LINEAR SEQUENCE ASSOCIATED TO SUM OF DIVISORS EVALUATED AT POLYNOMIAL ARGUMENTS

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Following the recent method of Deshouillers and the author in the theory of distribution modulo 1, we show that the sequence with general term $b_n = \sum_{m\le n}(m^2+1)/\sigma(m^2+1)$ is dense modulo 1.

1. INTRODUCTION

In 2010, J-M. Deshouillers and the author [**2**] introduced a method to prove density modulo 1 of the sequence $\Big(\sum_{m\le n}\dfrac{\varphi(m^2+1)}{m^2+1}\Big)_{n\ge1}$, where φ is the Euler function. We recall that a sequence of real numbers $(a_n)_{n\ge1}$ is said to be dense modulo 1 if the sequence of its fractional parts $(\{a_n\})_{n\ge1}$ is dense in the interval $[0,1)$. In this paper we study similar sequence defined over

$$s_m = \frac{m^2+1}{\sigma(m^2+1)},$$

where $\sigma(m)=\sum_{d|m} d$ is the sum of positive divisors of m. More precisely, we show the following result.

Theorem 1.1. *The sequence $(b_n)_{n\ge1}$ with general term defined by $b_n=\sum_{m\le n}s_m$ is dense modulo* 1.

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In comparison with the case of Euler function, we note that the function $\frac{\varphi(n)}{n}$ is strongly multiplicative (i.e., its values on the powers of the prime factor p of n depends only on p). But, the function $\frac{n}{\sigma(n)}$ is not strongly multiplicative.

2. SOME LEMMAS

To prove the theorem above we recall some lemmas. The working engine of the present paper is the following sieve result which is Proposition 2.1 in [**2**].

Lemma 2.2. *Assume that M is a sufficiently large integer. Let $\{\mathcal{P}_m\}_{1\le m\le M}$ be a family of finite and disjoint sets of primes p with $p \equiv 1 \pmod 4$ and $p > M^2+1$, and put $P_m = \prod_{p\in\mathcal{P}_m} p$. Then there exist infinitely many integers n such that for every integer $m = 1, 2, \ldots, M$ the integer $(n+m)^2+1$ is divisible by $(m^2+1)P_m$, and its prime factors that do not divide $(m^2+1)P_m$ are larger than $n^{\frac{1}{6M}}$.*

In order to make the lemma applicable to our method we need an appropriate family $\{\mathcal{P}_m\}_{1\le m\le M}$ of primes which fulfills a certain containment property. For the construction of this family we in turn require the following lemma and its corollary.

Lemma 2.3. *Assume that $(a_n)_{n\ge1}$ is a sequence of positive real numbers converging to 0 and that $\sum_{n=1}^{\infty} a_n = \infty$. Then the set of numbers $\sum_{n\in F} a_n$, where F runs over all finite subsets of $\mathbb{N}$, is dense in the positive real numbers.*

Proof. Let α be a positive real number. We construct a sequence $(F_k)_{k\ge1}$ of finite subsets of $\mathbb{N}$ with $\lim_{k\to\infty} \sum_{n\in F_k} a_n = \alpha$. Let $s_1 \ge 1$ be such that $a_{s_1} < \alpha$ and let $t_1 \ge s_1$ be maximal such that $\sum_{n=s_1}^{t_1} a_n < \alpha$. Put $F_1 = \{s_1, \ldots, t_1\}$. Assume that finite subsets $F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq \mathbb{N}$ are already defined such that $\sum_{n\in F_{k-1}} a_n < \alpha$. There exists an index $s_k > t_{k-1}$ such that $a_{s_k} < \alpha - \sum_{n\in F_{k-1}} a_n$. Choose $t_k \ge s_k$ maximal such that $\sum_{n=s_k}^{t_k} a_n < \alpha - \sum_{n\in F_{k-1}} a_n$ and put $F_k = F_{k-1} \cup \{s_k, \ldots, t_k\}$. Note that $\sum_{n\in F_k} a_n < \alpha \le \sum_{n\in F_k} a_n + a_{t_k+1}$, that is $0 < \alpha - \sum_{n\in F_k} a_n \le a_{t_k+1}$. As this tends to 0 with $k \to \infty$ we get the assertion of the Lemma.

Corollary 2.4. *For $N \ge 1$ let P_N be the set of primes $p \equiv 1 \pmod 4$ with $p \ge N$. Then, the set of numbers $\prod_{p\in F}\left(1 - \frac{1}{p}\right)$ where F runs over all finite subsets of P_N, is dense modulo 1.*

Proof. It is enough to prove that the set of numbers $-\sum_{p\in F}\log\left(1-\frac{1}{p}\right)$ where F runs over all finite subsets of P_N, is dense in the positive real numbers. This follows from the facts

$$-\sum_{p\in P_N}\log\left(1-\frac{1}{p}\right)=+\infty,\qquad \text{and}\qquad \lim_{\substack{p\to\infty\\ p\in P_N}}\log\left(1-\frac{1}{p}\right)=0,$$

and from Lemma 2.3. □

Also, we need some sharp bounds for the function $\sigma(n)$. It is known [**3**] that for $n\geq 7$ we have $\sigma(n)<2.59n\log\log n$. By an easy computation we can modify this bound for our purpose as below.

Lemma 2.5. *For $n\geq 2$ we have $n<\sigma(n)<2.6n\log\log(n+4)$.*

Lemma 2.6. *Assume that $\mathcal{A}$ is any non-empty subset of prime numbers p with the property $p>N\geq 1$. Then, we have*

$$1<\prod_{p\in\mathcal{A}}\left(1+\frac{1}{p^2-1}\right)<1+\frac{2}{N}.$$

Proof. Since for $x>0$ we have $\log(1+x)<x$, we imply

$$\begin{aligned}0<\log\prod_{p\in\mathcal{A}}\left(1+\frac{1}{p^2-1}\right)&=\sum_{p\in\mathcal{A}}\log\left(1+\frac{1}{p^2-1}\right)\\&<\sum_{p\in\mathcal{A}}\frac{1}{p^2-1}<\sum_{n=N+1}^{\infty}\frac{1}{n^2-1}=\frac{2N+1}{2N(N+1)}<\frac{1}{N}.\end{aligned}$$

Also, we note that $e^x<1+2x$ is valid for $0<x\leq 1$. This completes the proof. □

We need all the results above to obtain the following key lemma. It is a starting point of the proof of Theorem 1.1. We denote by $\mathbb{P}$ the set of primes $p\equiv 1 \pmod 4$.

Lemma 2.7. *Let $\delta>0$ be sufficiently small and put $M=\left\lfloor\frac{1}{\delta}\right\rfloor+1$. Then there exists a family $\{\mathcal{P}_m\}_{1\leq m\leq M}$ of pairwise disjoint finite sets of primes $p\equiv 1 \pmod 4$, all of which are $>M^2+1$ such that for $1\leq m\leq M$*

$$\frac{P_m}{\sigma(P_m)}\in\left(\frac{5\delta}{4s_m},\frac{7\delta}{4s_m}\right). \tag{2.1}$$

Proof. Note that $\frac{4}{2.6\log\log(M^2+1)}>7\delta$, and that $\frac{2}{M^2+1}<\frac{\delta}{8}$, for δ sufficiently small. By Lemma 2.5, for $1\leq m\leq M$ we have

$$4s_m>\frac{4}{2.6\log\log(m^2+5)}\geq\frac{4}{2.6\log\log(M^2+1)}>7\delta,$$

and therefore $\left(\frac{5\delta}{4s_m}, \frac{7\delta}{4s_m}\right) \subseteq (0,1)$. By Corollary 2.4, there is a finite set $\mathcal{P}_1 \subseteq \mathbb{P}$ of primes all of which are strictly larger than M^2+1 with

$$\left| \prod_{p\in\mathcal{P}_1}\left(1-\frac{1}{p}\right) - \frac{3\delta}{2s_1}\right| < \frac{1}{8s_1}.$$

Assume that the finite pairwise disjoint sets $\mathcal{P}_1, \ldots, \mathcal{P}_m \subseteq \mathbb{P}$ of primes are already chosen, and $m < M$. By Corollary 2.4, there is a finite set $\mathcal{P}_{m+1} \subseteq \mathbb{P} - (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m)$ of primes all of which are strictly larger than M^2+1 such that

$$\left| \prod_{p\in\mathcal{P}_{m+1}}\left(1-\frac{1}{p}\right) - \frac{3\delta}{2s_{m+1}}\right| < \frac{1}{8s_{m+1}}.$$

For $1 \le m \le M$ we have, say

$$\begin{aligned}\frac{P_m}{\sigma(P_m)} &= \prod_{p\in\mathcal{P}_m}\frac{p}{p+1} = \prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}+\frac{1}{p(p+1)}\right)\\ &= \prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}\right)\prod_{p\in\mathcal{P}_m}\left(1+\frac{1}{p^2-1}\right) = c(\mathcal{P}_m)\prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}\right).\end{aligned}$$

Note that, by Lemma 2.6, $c(\mathcal{P}_m) \in \left(1, 1+\frac{2}{M^2+1}\right)$ is a constant depending on $\mathcal{P}_m$. Recall that $P_m = \prod_{p\in\mathcal{P}_m} p$. Thus, for $1 \le m \le M$ we obtain

$$\begin{aligned}\left|\frac{P_m}{\sigma(P_m)} - \frac{3\delta}{2s_m}\right| &= \left|c(\mathcal{P}_m)\prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}\right) - \frac{3\delta}{2s_m}\right|\\ &\le \left|\prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}\right) - \frac{3\delta}{2s_m}\right| + \frac{2}{M^2+1} \le \frac{\delta}{8s_m} + \frac{\delta}{8} \le \frac{\delta}{4s_m}.\end{aligned}$$

This gives validity of the containment (2.1).

3. PROOF OF THEOREM 1.1

We take $\delta > 0$ to be a small positive number, and choose $M = \left\lfloor \frac{1}{\delta}\right\rfloor + 1$. For δ sufficiently small we apply Lemma 2.7 to end up with a family $\{\mathcal{P}_m\}_{1\le m\le M}$. By Lemma 2.2 there is an infinite set N of positive integers n such that for $n \in N$ and $1 \le m \le M$ we get $(m^2+1)P_m | (n+m)^2+1$ and for all primes p with $p|(n+m)^2+1$, $p \nmid (m^2+1)P_m$ we have $p > n^{1/6M}$. Hence

$$\prod_{\substack{p|(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\left(1-\frac{1}{p}\right) = \left(1+O\big(n^{-1/6M}\big)\right)^{\omega((n+m)^2+1)-\omega((m^2+1)P_m)},$$

where $\omega(n) = \sum_{p|n} 1$. Remember that $\omega(n) \ll \frac{\log n}{\log\log n}$. Thus, we obtain

$$\omega((n+m)^2+1) \ll \frac{\log(n+m)}{\log\log(n+m)} \leq \frac{\log(n+M)}{\log\log n}, \qquad (\text{as } n \to \infty).$$

We put $f_\alpha(p) = \frac{p^\alpha}{\sigma(p^\alpha)} = 1 - \frac{1}{p} + O(p^{-2})$, where the O-constant does not depend on α. For $n \in N$, $n \to \infty$ we conclude

$$\prod_{\substack{p|(n+m)^2+1 \\ p\nmid(m^2+1)P_m}} \left(1 - \frac{1}{p}\right) = 1 + O\left(\frac{\log(n+m)}{n^{1/6M}\log\log n}\right) = 1 + o(1).$$

For every prime divisor p of $(n{+}m)^2{+}1$ let α_p be maximal such that $p^{\alpha_p}|(n{+}m)^2{+}1$. We get

$$\prod_{\substack{p|(n+m)^2+1 \\ p\nmid(m^2+1)P_m}} f_{\alpha_p}(p) = 1 + o(1), \qquad (\text{for } n \in N, n \to \infty). \tag{3.2}$$

Remember that $(m^2+1)P_m|(n+m)^2+1$. $\gcd\big((m^2+1), P_m\big) = 1$ implies

$$\prod_{p|(m^2+1)P_m} f_{\alpha_p}(p) = \frac{m^2+1}{\sigma(m^2+1)}\frac{P_m}{\sigma(P_m)} = s_m\frac{P_m}{\sigma(P_m)}.$$

This results in

$$\begin{aligned} s_{n+m} &= \prod_{p|(n+m)^2+1} f_{\alpha_p}(p) \\ &= \prod_{p|(m^2+1)P_m} f_{\alpha_p}(p) \prod_{\substack{p|(n+m)^2+1 \\ p\nmid(m^2+1)P_m}} f_{\alpha_p}(p) = s_m\frac{P_m}{\sigma(P_m)}(1+o(1)) \in \left(\delta, \frac{9}{5}\delta\right) \end{aligned}$$

for $1 \leq m \leq M$ and $n \in N$, n large enough. We obtain $1 \leq M\delta \leq \sum_{m=1}^{M} s_{n+m} < \frac{9}{5}M\delta \leq \frac{9}{5}(\delta+1) < \frac{9}{5} + 2\delta < 2$. Remember that $b_n = \sum_{k=1}^{n}\frac{k^2+1}{\sigma(k^2+1)}$. Hence $b_{n+M} - b_n = \sum_{m=1}^{M} s_{n+m}$. We get $1 \leq b_{n+M} - b_n < 2$. Therefore we have $\lfloor b_n \rfloor + 1 < b_{n+M} < \lfloor b_n \rfloor + 3$. We distinguish two cases:

Case 1. Assume that $\lfloor b_n \rfloor + 2 \leq b_{n+M}$. Then we take u such that $b_{n+u} < \lfloor b_n \rfloor + 1 \leq b_{n+u+1}$, and v such that $b_{n+v} < \lfloor b_n \rfloor + 2 \leq b_{n+v+1}$. So, we have

$\lfloor b_n \rfloor + 1 \leq b_{n+u+1} < \cdots < b_{n+v} < \lfloor b_n \rfloor + 2$. By reducing all terms $b_{n+u+1}, \ldots, b_{n+v}$ modulo 1, we obtain $0 \leq \{b_{n+u+1}\} < \cdots < \{b_{n+v}\} < 1$, with

$$\max \left\{ \{b_{n+u+1}\}, 1 - \{b_{n+v}\}, \max_{u+2 \leq m \leq v} \left(\{b_{n+m}\} - \{b_{n+m-1}\} \right) \right\} < 2\delta.$$

This implies that for each subinterval I of $[0,1]$ with length larger than 2δ there exist $i \in \{u+1, \ldots, v\}$ such that $\{b_{n+i}\} \in I$.

Case 2. Assume that $b_{n+M} < \lfloor b_n \rfloor + 2$. We set u such that $b_{n+u} < \lfloor b_n \rfloor + 1 \leq b_{n+u+1}$. Also, we let v such that $b_{n+v} < b_n + 1$. This is possible, because $b_{n+M} - b_n > 1$. If we reduce all terms $b_n, \ldots, b_{n+v}$ modulo 1, then we obtain

$$0 \leq \{b_{n+u+1}\} < \cdots < \{b_{n+v}\} < \{b_n\} < \cdots < \{b_{n+u}\} < 1,$$

with $\max \{\{b_{n+u+1}\}, 1 - \{b_{n+u}\}, (\{b_n\} - \{b_{n+v}\})\} < 2\delta$, and also

$$\max \left\{ \max_{u+2 \leq m \leq v} \left(\{b_{n+m}\} - \{b_{n+m-1}\} \right), \max_{1 \leq m \leq u} \left(\{b_{n+m}\} - \{b_{n+m-1}\} \right) \right\} < 2\delta.$$

Again, we imply that for each subinterval I of $[0,1]$ with length larger than 2δ there exists $i \in \{0, \ldots, v\}$ such that $\{b_{n+i}\} \in I$.

In both of above cases, since $\delta > 0$ was arbitrary small, we get our desired density result.

4. SOME OBSERVATIONS ON THE DISTRIBUTION OF THE VALUES OF b_n

Let us study the distribution of the sequence b_n modulo 1. In Figure 1 we have pictured the pointset $(n, \{b_n\})$ for $1 \leq n \leq 1000$. It strikes us that there is some pattern cognoscible. Our first question refers to the mathematical background of this pattern.

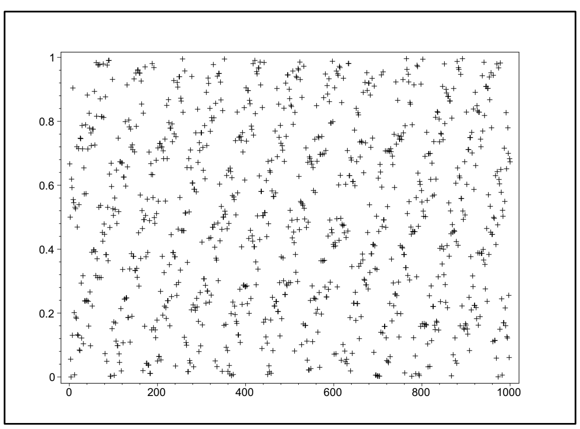


Figure 1. Graph of the pointset $(n, \{b_n\})$ for $1 \leq n \leq 1000$

It may very well be that the sequence b_n is not only dense but even uniformly distributed modulo 1. This means that for every subinterval $[a,b]$ of $I=[0,1]$

$$\lim_{N\to\infty}\frac{1}{N}\#\Big\{n\leq N:\{b_n\}\in[a,b]\Big\}=b-a.$$

The following criterion of WEYL [**4**] allows us to characterizes uniform distribution modulo 1 of a given sequence.

Theorem 3.8 (Weyl criterion - 1914)**.** *The sequence* $\{a_n\}_{n=1}^{\infty}$ *is uniformly distributed modulo* 1 *if and only if, for every positive integer* h *we have*

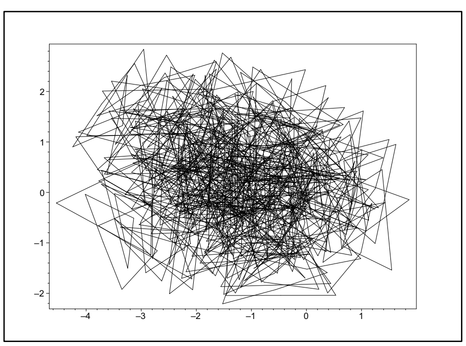
$$\sum_{n\leq N}e(ha_n)=o(N),$$

as N *tends to infinity. Here* $e(x)=e^{2\pi ix}$.

In 1981, F. DEKKING and M. MENDÈS FRANCE [**1**] introduced the idea of making visible the Weyl sums $\sum_{n\leq N}e(ha_n)$ for a given real sequence a_n and given positive integer h. Indeed, for given $h,N\in\mathbb{N}$ they draw in $\mathbb{R}^2$ a plane curve generated by successively connected lines segment, which joint the point V_n to V_{n+1} with

$$V_n=\left(\sum_{k=1}^{n}\cos(2\pi ha_k),\sum_{k=1}^{n}\sin(2\pi ha_k)\right),$$

for $1\leq n\leq N$.



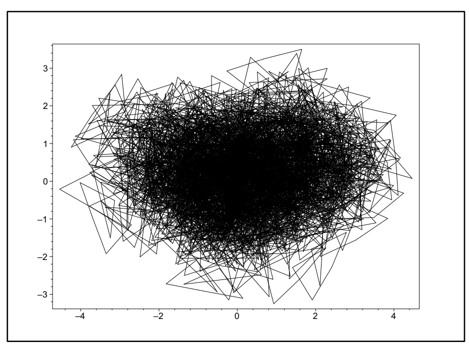


Figure 2. Graph of Weyl sums $\sum_{n\leq 1000}e(b_n)$ and $\sum_{n\leq 5000}e(b_n)$, respectively left and right

Note that the length of each line segment is 1. Thus, if $1\leq n\leq N$, then the frame that includes the Dekking - Mendès France curve has the size not exceeding $N\times N$, and geometrically, the Weyl criterion asserts that the related sequence is uniformly distributed modulo 1 if and only if height and width of the frame $=o(N)$ as N tends to infinity. Figure 2 shows the Dekking - Mendès France curve of the Weyl sums $\sum_{n\leq 1000} e(b_n)$ and $\sum_{n\leq 5000} e(b_n)$. By considering very small frames we conjecture that the sequence is uniformly distributed modulo 1.

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