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FRACTIONAL DERIVATIVES OF COLOMBEAU GENERALIZED STOCHASTIC PROCESSES DEFINED ON $\mathbb{R}^+$

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We consider Caputo and Riemann-Liouville fractional derivatives of a Colombeau generalized stochastic process G defined on $\mathbb{R}^+$. We give proper definitions and prove that both are Colombeau generalized stochastic processes themselves. We also give a solution to a certain Cauchy problem illustrating the application of the theory.

1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The concept involving fractional derivatives is successfully used for modeling phenomena that appear very often in applications, especially in physics, meteorology, climatology, hydrology, geophysics, economy. Fractional processes have also witnessed an increasing development in the last decade. For instance, they are convenient for describing the long memory properties of many time series. For more about fractional processes we refer, for instance, to [**1**], [**3**] and [**7**].

Colombeau generalized stochastic process are introduced to make possible some nonlinear stochastic problems (see [**6**], [**9**], [**10**] and [**11**]). For more about the Colombeau theory in general we refer to [**2**] and [**8**].

Fractional derivatives of Colombeau generalized stochastic processes defined on $[0, \infty)$ are introduced in [**14**] where it is proved that a Caputo fractional derivative of a Colombeau generalized stochastic process G is a Colombeau generalized

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stochastic process itself only if G satisfies certain conditions. One of the possible approaches to get rid of these restrictions is to make a regularization of the fractional derivative, as done in [**14**]. Some other possibilities are considered in [**13**]. In this paper, we avoid the problematic zero and consider fractional derivatives of Colombeau generalized stochastic process on the interval $(0,\infty)$ instead of $[0,\infty)$.

The paper is organized as follows. After the introduction, in the second section we give some basic preliminaries such as notation and definitions of the objects we shall work with. We also introduce different spaces of Colombeau generalized stochastic processes.

In the third section we introduce fractional derivatives od Colombeau generalized stochastic process defined on $[0,\infty)$ by briefly exposing some results from [**14**].

The fourth section of the paper is devoted to the Caputo fractional derivative of a Colombeau generalized stochastic process defined on $(0,\infty)$. We prove that it is a Colombeau generalized stochastic process itself. This fact represents the adventage of this approach in comparison to the one where we consider Colombeau generalized stochastic processes on the interval $[0,\infty)$.

Finally, in the fifth section we introduce the Riemann-Liouville fractional derivative of a Colombeau generalized stochastic process. We show that, if the Colombeau generalized stochastic process G is defined on $[0,\infty)$, by defining the Riemann-Liouville fractional derivative of G we do not gain much, since it is a Colombeau generalized stochastic process itself only if all derivatives of G_ε are vanishing at zero which is, as known from the classical fractional derivative theory, exactly the case when the Riemann-Liouville fractional derivative coincides with the Caputo fractional derivative. On the other hand, the αth Riemann-Liouville fractional derivative of a Colomebau generalized stochastic process G defined on $(0,\infty)$ is a Colombeau generalized stochastic process itself, for every α, with no restrictions on G. At the end of the fifth section we give the solution of a certain Cauchy problem that illustrates the application of the theory.

2. PRELIMINARIES

Let (Ω,Σ,μ) be a probability space. Denote by $\mathcal{D}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$. Generalized stochastic process on $\mathbb{R}$ is a weakly measurable mapping $X:\Omega\to\mathcal{D}'(\mathbb{R})$. We denote by $\mathcal{D}'_\Omega(\mathbb{R})$ the space of generalized stochastic processes. For each fixed function $\varphi\in\mathcal{D}(\mathbb{R})$, the mapping $\Omega\to\mathbb{R}$ defined by $\omega\mapsto\langle X(\omega),\varphi\rangle$ is a random variable. For instance, the white noise $\dot{W}:\Omega\to\mathcal{D}'(\mathbb{R})$ is the identity mapping $\dot{W}(\omega)=\omega$, i.e., $\langle\dot{W}(\omega),\varphi\rangle=\langle\omega,\varphi\rangle$, $\varphi\in\mathcal{D}(\mathbb{R})$. (For more details see [**4**]) A net φ_ε of mollifiers given by $\varphi_\varepsilon(t)=\frac{1}{\varepsilon}\varphi\Big(\frac{t}{\varepsilon}\Big)$, $\varphi\in\mathcal{D}(\mathbb{R})$, $\int\varphi(t)\mathrm{d}t=1$, is called a nonnegative model delta net. Smoothed white noise process on $\mathbb{R}$ is defined as

$$\dot{W}_\varepsilon(t)=\langle\dot{W}(s),\varphi_\varepsilon(t-s)\rangle, \tag{1}$$

where $\dot{W}$ is the white noise process on $\mathbb{R}$ and φ_ε is a nonnegative model delta net.

In the sequel we introduce Colombeau generalized stochastic processes as done in [**9**] and [**10**]. (For some other possible approaches to generalized stochastic processes see, e.g. [**5**] and [**12**]). We confine ourselves to the one-dimensional case.

Denote by $\mathcal{E}^{\Omega}((0,\infty))$ the space of nets $(X_\varepsilon)_{\varepsilon\in(0,1)} = (X_\varepsilon)_\varepsilon$, of stochastic processes X_ε with almost surely continuous paths, i.e., the space of nets of processes $X_\varepsilon : (0,1)\times(0,\infty)\times\Omega \to \mathbb{R}$ such that

$$(t,\omega)\mapsto X_\varepsilon(t,\omega) \text{ is jointly measurable, for all } \varepsilon\in(0,1);$$
$$t\mapsto X_\varepsilon(t,\omega) \text{ belongs to } \mathcal{C}^\infty((0,\infty)), \text{ for all } \varepsilon\in(0,1) \text{ and almost all } \omega\in\Omega.$$

By $\mathcal{E}^{\Omega}_M((0,\infty))$ we denote the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}^{\Omega}((0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$, there exist constants $N,\ C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[\varepsilon,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ has a moderate bound, i.e., $\sup_{t\in[\varepsilon,T]}|\partial^\alpha X_\varepsilon(t,\omega)| \le C\,\varepsilon^{-N}$, $\varepsilon\le\varepsilon_0$.

$\mathcal{N}^{\Omega}((0,\infty))$ is the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}^{\Omega}_M((0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$ and all $b\in\mathbb{R}$, there exist constants $C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[\varepsilon,T]}|\partial^\alpha X_\varepsilon(t,\omega)| \le C\,\varepsilon^{b}$, $\varepsilon\le\varepsilon_0$. Then we say that $\sup_{t\in[\varepsilon,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ is negligible.

Then $\mathcal{G}^{\Omega}((0,\infty)) = \mathcal{E}^{\Omega}_M((0,\infty))/\mathcal{N}^{\Omega}((0,\infty))$ is a differential algebra (differentiation with respect to the variable t and pointwise multiplication) called *algebra of Colombeau generalized stochastic processes*. The elements of $\mathcal{G}^{\Omega}((0,\infty))$ will be denoted by $X=[X_\varepsilon]$, where $(X_\varepsilon)_\varepsilon$ is a representative of the class.

Every stochastic process can be viewed as Colombeau generalized stochastic process. It follows from the usual embedding arguments of Colombeau theory (see [**8**]). For instance, the Colombeau generalized white noise process has the representative given by (1).

Instead of the interval $(0,\infty)$ one can consider the interval $[0,\infty)$ (or any other interval in $\mathbb{R}$) and similarly define the algebra $\mathcal{G}^{\Omega}([0,\infty))$ as done in [**14**]. Here we will recall those definitions:

Denote by $\mathcal{E}^{\Omega}([0,\infty))$ the space of nets $(X_\varepsilon)_{\varepsilon\in(0,1)} = (X_\varepsilon)_\varepsilon$, of stochastic processes X_ε with almost surely smooth paths, i.e., the space of nets of processes $X_\varepsilon : (0,1)\times[0,\infty)\times\Omega \to \mathbb{R}$ such that

$$(t,\omega)\mapsto X_\varepsilon(t,\omega) \text{ is jointly measurable, for all } \varepsilon\in(0,1);$$
$$t\mapsto X_\varepsilon(t,\omega) \text{ belongs to } \mathcal{C}^\infty([0,\infty)) \text{ for all } \varepsilon\in(0,1) \text{ and almost all } \omega\in\Omega.$$

By $\mathcal{E}^{\Omega}_M([0,\infty))$ we denote the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}^{\Omega}([0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$, there exist constants $N,\ C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ has a moderate bound, i.e., $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)| \le C\,\varepsilon^{-N}$, $\varepsilon\le\varepsilon_0$.

$\mathcal{N}^{\Omega}([0,\infty))$ is the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}^{\Omega}_M([0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$ and all $b\in\mathbb{R}$, there exist constants $C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)| \le C\,\varepsilon^{b}$, $\varepsilon\le\varepsilon_0$. Then we say that $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ is negligible.

Then $\mathcal{G}^{\Omega}([0,\infty)) = \mathcal{E}_M^{\Omega}([0,\infty))/\mathcal{N}^{\Omega}([0,\infty))$ is a corresponding algebra of Colombeau generalized stochastic processes.

We also introduce so-called $\mathcal{C}^k$-Colombeau generalized stochastic processes.

Let $k \in \mathbb{N}$. Denote by $\mathcal{E}_{M,\mathcal{C}^k}^{\Omega}([0,\infty))$ the space of nets of continuous processes $(X_\varepsilon)_\varepsilon$ on $[0,\infty)$, with the property that for almost all $\omega \in \Omega$ and for all $T > 0$, there exist constants $N,\ C > 0$ and $\varepsilon_0 \in (0,1)$ such that $\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)|$ has a moderate bound for $m \in \{0,\dots,k\}$, i.e., $\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)| \le C\ \varepsilon^{-N}$, $m \in \{0,\dots,k\}$, $\varepsilon \le \varepsilon_0$.

Let $k \in \mathbb{N}$. $\mathcal{N}_{\mathcal{C}^k}^{\Omega}([0,\infty))$ is the space of nets of processes $(X_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathcal{C}^k}^{\Omega}([0,\infty))$, with the property that for almost all $\omega \in \Omega$ and for all $T > 0$ and all $b \in \mathbb{R}$, there exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that $\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)| \le C\ \varepsilon^{b}$, $m \in \{0,\dots,k\}$, $\varepsilon \le \varepsilon_0$. Then we say that $\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)|$ is negligible for $m \in \{0,\dots,k\}$.

Then, for $k \in \mathbb{N}$, the factor space $\mathcal{G}_{\mathcal{C}^k}^{\Omega}([0,\infty)) = \mathcal{E}_{M,\mathcal{C}^k}^{\Omega}([0,\infty))/\mathcal{N}_{\mathcal{C}^k}^{\Omega}([0,\infty))$ is an algebra and it is called *algebra of $\mathcal{C}^k$-Colombeau generalized stochastic processes.*

Finally, for evaluation of generalized stochastic process at fixed points of time, we introduce the concept of a Colombeau generalized random variable as follows. Let $\mathcal{E}R$ be the space of nets of measurable functions on Ω.

$\mathcal{E}R_M$ is the space of nets $(X_\varepsilon)_\varepsilon \in \mathcal{E}R$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$ there exist constants $N,\ C > 0$, and $\varepsilon_0 \in (0,1)$ such that $|X_\varepsilon(\omega)| \le C\ \varepsilon^{-N}$, $\varepsilon \le \varepsilon_0$.

$\mathcal{N}R$ is the space of nets $(X_\varepsilon)_\varepsilon \in \mathcal{E}R$, $\varepsilon \in (0,1)$, with the property that for almost all $\omega \in \Omega$ and all $b \in \mathbb{R}$, there exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that $|X_\varepsilon(\omega)| \le C\ \varepsilon^{b}$, $\varepsilon \le \varepsilon_0$.

The differential algebra $\mathcal{G}R$ of Colombeau generalized random variables is the factor algebra $\mathcal{G}R = \mathcal{E}R_M/\mathcal{N}R$.

3. SOME BASICS ON FRACTIONAL DERIVATIVES OF COLOMBEAU GENERALIZED STOCHASTIC PROCESSES DEFINED ON $[0,\infty)$

In [**14**] the Caputo αth fractional derivative and the regularized Caputo αth fractional derivative of a Colombeau generalized stochastic process defined on $[0,\infty)$ are introduced. Here we recall some definitions and basic properties.

Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^{\Omega}([0,\infty))$. The Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, is defined by

$$
{}_0^cD_t^\alpha G_\varepsilon(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, \mathrm{d}\tau, & m-1 < \alpha < m \\ G_\varepsilon^{(m)}(t) = \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\, G_\varepsilon(t), & \alpha = m \end{cases}, \tag{2}
$$

for $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$.

For $m-1<\alpha<m$, $m\in\mathbb{N}$, by using a simple change of variables one obtains

$$
{}_0^cD_t^\alpha G_\varepsilon(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{G_\varepsilon^{(m)}(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s.
$$

For the proof of the following proposition we refer to [**14**].

Proposition 1. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$ and let the Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha>0$, be given by* (2). *Then, for every $\alpha>0$, $\sup_{t\in[0,T]}|\,{}_0^cD_t^\alpha G_\varepsilon(t)|$ has a moderate bound.*

Let $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ be two representatives of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$. Then, $\sup_{t\in[0,T]}|\,{}_0^cD_t^\alpha G_{1\varepsilon}(t)-{}_0^cD_t^\alpha G_{2\varepsilon}(t)|$ is negligible, for every $\alpha>0$.

According to the Proposition 1 the Caputo αth fractional derivative of a Colombeau generalized stochastic process on $[0,\infty)$ can be defined as an element of $\mathcal{G}^\Omega_{C^0}([0,\infty))$.

Definition 1. *Let $G(t)\in\mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process on $[0,\infty)$. The Caputo αth fractional derivative of $G(t)$, in notation ${}_0^cD_t^\alpha G(t)=[({}_0^cD_t^\alpha G_\varepsilon(t))_\varepsilon]$, $\alpha>0$, is an element of $\mathcal{G}^\Omega_{C^0}([0,\infty))$ satisfying* (2).

Note that, in general, the first order derivative of ${}_0^cD_t^\alpha G_\varepsilon(t)$, $\frac{\mathrm{d}}{\mathrm{d}t}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, for $m-1<\alpha<m$, $m\in\mathbb{N}$, is not defined at the point $t=0$. Indeed,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,{}_0^cD_t^\alpha G_\varepsilon(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{G_\varepsilon^{(m)}(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s\right]\\
&=\frac{1}{\Gamma(m-\alpha)}\left[\int_0^t\frac{G_\varepsilon^{(m+1)}(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s+\frac{G_\varepsilon^{(m)}(0)}{t^{\alpha+1-m}}\right]
\end{aligned}
$$

which is not defined in zero, unless $G_\varepsilon^{(m)}(0)=0$. In order to have the second order derivative $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, $m-1<\alpha<m$, $m\in\mathbb{N}$, defined on the whole interval $[0,\infty)$, one additionally needs the condition $G_\varepsilon^{(m+1)}(0)=0$. In general, the kth order derivative $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, $m-1<\alpha<m$, $m,k\in\mathbb{N}$, is defined on the whole interval $[0,\infty)$, if $G_\varepsilon^{(m+l)}(0)=0$, for all $l=0,\dots,k-1$.

The following assertion holds (for the details of the proof, see [**14**]).

Theorem 1. *Let $G(t)\in\mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process on $[0,\infty)$. The Caputo αth fractional derivative ${}_0^cD_t^\alpha G(t)$ is a Colombeau generalized stochastic process (an element of $\mathcal{G}^\Omega([0,\infty))$) for $m-1<\alpha<m$, $m\in\mathbb{N}$, if $G_\varepsilon^{(m)}(0)=G_\varepsilon^{(m+1)}(0)=G_\varepsilon^{(m+2)}(0)=\cdots=0$.*

Moreover, if $G_\varepsilon^{(m)}(0)=0$, for every $m=1,2,\dots,$ then, for every $\alpha>0$, the Caputo αth fractional derivative ${}_0^cD_t^\alpha G(t)$ is a Colombeau generalized stochastic process, i.e., an element of $\mathcal{G}^\Omega([0,\infty))$.

The previous theorem illustrates that a Caputo fractional derivative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^{\Omega}([0,\infty))$ is a Colombeau generalized stochastic process itself only if G satisfies certain conditions. If one wants this to be satisfied for an arbitrary $G(t) \in \mathcal{G}^{\Omega}([0,\infty))$, one of the possible approaches is to make a regularization of the fractional derivative, as in [**14**].

4. CAPUTO FRACTIONAL DERIVATIVES OF COLOMBEAU PROCESSES DEFINED ON $\mathbb{R}^+$ INSTEAD OF $[0,\infty)$

Here we introduce the Caputo αth fractional derivative of a Colombeau generalized stochastic process defined on $(0,\infty)$ instead of $[0,\infty)$. As we will see, it is a Colombeau generalized stochastic process itself, for every α, with no restrictions on G.

Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^{\Omega}((0,\infty))$. The Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, is defined by

$$(3) \qquad {}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^{t-\varepsilon} \dfrac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, \mathrm{d}\tau, & m-1<\alpha<m \\ G_\varepsilon^{(m)}(t) = \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\, G_\varepsilon(t), & \alpha = m \end{cases},$$

for $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$.

Remark 1. Note that, for $m-1<\alpha<m$, $m \in \mathbb{N}$,

$${}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^{t-\varepsilon} \frac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, \mathrm{d}\tau = \frac{1}{\Gamma(m-\alpha)} \int_\varepsilon^{t} \frac{G_\varepsilon^{(m)}(t-s)}{s^{\alpha+1-m}}\, \mathrm{d}s.$$

The last equality is easily obtained by using a simple change of variables.

Lemma 1. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^{\Omega}((0,\infty))$. Let the Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, ${}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t)$, $\alpha > 0$, be given by the relation* (3). *Then, for every $\alpha > 0$, we have $({}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t))_\varepsilon \in \mathcal{E}_M^{\Omega}((0,\infty))$.*

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. First, note that for $\alpha \in \mathbb{N}$, ${}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t)$ is the usual derivative of order α of $G_\varepsilon(t)$ and since $(G_\varepsilon(t))_\varepsilon \in \mathcal{E}_M^{\Omega}((0,\infty))$ the assertion immediately follows.

Now, consider the case when $m-1<\alpha<m$, $m \in \mathbb{N}$. Then, we have

$$\begin{aligned} \sup_{t\in[\varepsilon,T]} |{}_0^cD_{t-\varepsilon}^{\alpha} G_\varepsilon(t)| &\le \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[\varepsilon,T]} \int_0^{t-\varepsilon} \left| \frac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, \mathrm{d}\tau \right| \\ &\le \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[\varepsilon,T]} \int_0^{t-\varepsilon} \frac{\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}} \\ &= \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[\varepsilon,T]} \int_\varepsilon^{t} \frac{\mathrm{d}s}{s^{\alpha+1-m}} \end{aligned}$$

$$= \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[\varepsilon,T]} \frac{t^{m-\alpha}-\varepsilon^{m-\alpha}}{m-\alpha}$$
$$\le \frac{1}{\Gamma(m-\alpha)} \frac{T^{m-\alpha}-\varepsilon^{m-\alpha}}{m-\alpha} \sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m)}(\tau)|.$$

Since $G(t) \in \mathcal{G}^\Omega((0,\infty))$, it follows that $\sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m)}(\tau)|$ has a moderate bound. Therefore, $\sup_{t\in[\varepsilon,T]} |\,{}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)|$ has a moderate bound, for every $\alpha > 0$.

To obtain a moderate bound for the first order derivative $\frac{\mathrm{d}}{\mathrm{d}t}\big({}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)\big)$ first note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big({}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)\big) = \frac{1}{\Gamma(m-\alpha)}\left[\int_\varepsilon^t \frac{G_\varepsilon^{(m+1)}(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s + \frac{G_\varepsilon^{(m)}(0)}{t^{\alpha+1-m}}\right].$$

Now, we have

$$\sup_{t\in[\varepsilon,T]}\left|\frac{\mathrm{d}}{\mathrm{d}t}\big({}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)\big)\right|$$
$$\le \frac{1}{\Gamma(m-\alpha)}\left[\sup_{\tau\in[\varepsilon,T]}|G_\varepsilon^{(m+1)}(\tau)| \sup_{t\in[\varepsilon,T]}\int_\varepsilon^t \frac{\mathrm{d}s}{s^{\alpha+1-m}} + \sup_{t\in[\varepsilon,T]}\frac{G_\varepsilon^{(m)}(0)}{t^{\alpha+1-m}}\right]$$
$$\le \frac{1}{\Gamma(m-\alpha)}\left[\sup_{\tau\in[\varepsilon,T]}|G_\varepsilon^{(m+1)}(\tau)| \frac{T^{m-\alpha}-\varepsilon^{m-\alpha}}{m-\alpha} + \frac{G_\varepsilon^{(m)}(0)}{\varepsilon^{\alpha+1-m}}\right].$$

Using the same argument as above one obtains that $\sup_{t\in[\varepsilon,T]}\left|\frac{\mathrm{d}}{\mathrm{d}t}\big({}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)\big)\right|$ has a moderate bound. Higher order derivatives can be estimated in a similar way. Thus, $({}^c_0D^\alpha_{t-\varepsilon} G_\varepsilon(t))_\varepsilon \in \mathcal{E}^\Omega_M((0,\infty))$, for every $\alpha > 0$. $\square$

The following lemma shows that for two representatives of a Colombeau generalized stochastic process from $\mathcal{G}^\Omega((0,\infty))$ the difference of their Caputo αth fractional derivatives is an element of $\mathcal{N}^\Omega((0,\infty))$.

Lemma 2. *Let $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ be two different representatives of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$. Then, for every $\alpha > 0$, we have $({}^c_0D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t))_\varepsilon - ({}^c_0D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t))_\varepsilon \in \mathcal{N}^\Omega((0,\infty))$.*

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. First, note that for $\alpha \in \mathbb{N}$, ${}^c_0D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t)$ and ${}^c_0D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t)$ are the usual derivatives of order α and since they represent the same Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$ it follows that $({}^c_0D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t))_\varepsilon - ({}^c_0D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t))_\varepsilon \in \mathcal{N}^\Omega((0,\infty))$ and the assertion immediately follows.

Now, consider the case when $m-1 < \alpha < m$, $m \in \mathbb{N}$. Then, we have

$$\sup_{t\in[\varepsilon,T]} |\,{}^c_0D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t) - {}^c_0D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t)|$$
$$\le \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[\varepsilon,T]} \int_\varepsilon^t \left|\frac{G_{1\varepsilon}^{(m)}(t-s) - G_{2\varepsilon}^{(m)}(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s\right|$$

$$\leq \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[\varepsilon,T]} |G_{1\varepsilon}^{(m)}(\tau) - G_{2\varepsilon}^{(m)}(\tau)| \sup_{t\in[\varepsilon,T]} \int_\varepsilon^t \frac{\mathrm{d}s}{s^{\alpha+1-m}}$$
$$\leq \frac{1}{\Gamma(m-\alpha)} \frac{T^{m-\alpha}-\varepsilon^{m-\alpha}}{m-\alpha} \sup_{\tau\in[\varepsilon,T]} |G_{1\varepsilon}^{(m)}(\tau) - G_{2\varepsilon}^{(m)}(\tau)|.$$

Since $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ are the representatives of $G(t) \in \mathcal{G}^\Omega((0,\infty))$ it follows that $\sup_{\tau\in[\varepsilon,T]} |G_{1\varepsilon}^{(m)}(\tau) - G_{2\varepsilon}^{(m)}(\tau)|$ is negligible. Therefore, one concludes that $\sup_{t\in[\varepsilon,T]} |\, {}_0^cD_{t-\varepsilon}^\alpha G_{1\varepsilon}(t) - {}_0^cD_{t-\varepsilon}^\alpha G_{2\varepsilon}(t)|$ is negligible, too.

For estimating the first derivative we do as follows

$$\sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left({}_0^cD_{t-\varepsilon}^\alpha G_{1\varepsilon}(t) - {}_0^cD_{t-\varepsilon}^\alpha G_{2\varepsilon}(t) \right) \right|$$
$$\leq \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[\varepsilon,T]} |G_{1\varepsilon}^{(m+1)}(\tau) - G_{2\varepsilon}^{(m+1)}(\tau)| \sup_{t\in[\varepsilon,T]} \int_\varepsilon^t \frac{\mathrm{d}s}{s^{\alpha+1-m}}$$
$$+ \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[\varepsilon,T]} \frac{G_{1\varepsilon}^{(m)}(0) - G_{2\varepsilon}^{(m)}(0)}{t^{\alpha+1-m}}$$
$$\leq \frac{1}{\Gamma(m-\alpha)} \left[\sup_{\tau\in[\varepsilon,T]} |G_\varepsilon^{(m+1)}(\tau)| \, \frac{T^{m-\alpha}-\varepsilon^{m-\alpha}}{m-\alpha} + \frac{G_{1\varepsilon}^{(m)}(0) - G_{2\varepsilon}^{(m)}(0)}{\varepsilon^{\alpha+1-m}} \right].$$

The negligibleness of the $\sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left({}_0^cD_{t-\varepsilon}^\alpha G_{1\varepsilon}(t) - {}_0^cD_{t-\varepsilon}^\alpha G_{2\varepsilon}(t) \right) \right|$ follows by using the similar argument. Higher order derivatives can be estimated in a similar way. Thus, for every $\alpha > 0$, $({}_0^cD_{t-\varepsilon}^\alpha G_{1\varepsilon}(t))_\varepsilon - ({}_0^cD_{t-\varepsilon}^\alpha G_{2\varepsilon}(t))_\varepsilon \in \mathcal{N}^\Omega((0,\infty))$. □

Now we are in the position to introduce the Caputo αth fractional derivative of a Colombeau generalized stochastic process on $(0,\infty)$.

Definition 2. *Let $G(t) \in \mathcal{G}^\Omega((0,\infty))$ be a Colombeau generalized stochastic process on $(0,\infty)$. The Caputo αth fractional derivative of $G(t)$, in notation ${}_0^cD_{t-\varepsilon}^\alpha G(t) = \left[\left({}_0^cD_{t-\varepsilon}^\alpha G_\varepsilon(t) \right)_\varepsilon \right]$, $\alpha > 0$, is an element of $\mathcal{G}^\Omega((0,\infty))$ satisfying* (3).

Now we denote $\tilde{G} = GH$, where G is continuous on $[0,\infty)$ having right derivatives in zero: $G(0+), G'(0+), \ldots G^{(m)}(0+)$ and H denotes the Heaviside function. The following relation is well known

$$(\tilde{GH})^{(m)}(t) = G^{(m)}(t)H(t) + \sum_{i=0}^{m-1} G^{(i)}(0)\delta^{(m-1-i)}(t), \tag{4}$$

where we used tilde notation for the distributional derivative. We define

$$\phi_\alpha(t) = \begin{cases} \dfrac{t^\alpha}{\Gamma(\alpha)}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and by using the Remark 1, for $m-1 < \alpha < m$, $m \in \mathbb{N}$, one obtains that

$${}_0^cD_{t-\varepsilon}^\alpha (\tilde{G} * \varphi_\varepsilon)(t) = \kappa_\varepsilon(t)\phi_{m-\alpha}(t) * (GH * \varphi_\varepsilon)^{(m)}(t),$$

and

$$\Big(({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}\tilde{G}) * \varphi_\varepsilon \Big)(t) = \kappa_\varepsilon(t)\phi_{m-\alpha}(t) * G^{(m)}(t)H(t) * \varphi_\varepsilon(t),$$

where

$$\kappa_\varepsilon(t) = \begin{cases} 0, & t < \varepsilon \\ 1, & t \geq \varepsilon \end{cases}$$

and the mollifier φ_ε above is chosen to have support in $[0,\infty)$: $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\,\varphi\Big(\frac{x-\varepsilon}{\varepsilon}\Big)$.

Now, by using the relations above and (4) one obtains

$$\begin{aligned}
{}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}(\tilde{G} * \varphi_\varepsilon) &= \kappa_\varepsilon \phi_{m-\alpha} * (GH * \varphi_\varepsilon)^{(m)} = \kappa_\varepsilon \phi_{m-\alpha} * (GH)^{(m)} * \varphi_\varepsilon \\
&= \kappa_\varepsilon \phi_{m-\alpha} * G^{(m)} H * \varphi_\varepsilon + \sum_{i=0}^{m-1} G^{(i)}(0) \Big(\kappa_\varepsilon \phi_{m-\alpha} * \varphi_\varepsilon * \delta^{(m-1-i)} \Big) \\
&= ({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}\tilde{G}) * \varphi_\varepsilon + \sum_{i=0}^{m-1} G^{(i)}(0) \left(\kappa_\varepsilon \phi_{m-\alpha} * \varphi_\varepsilon * \delta \right)^{(m-1-i)} \\
&= ({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}\tilde{G}) * \varphi_\varepsilon + \sum_{i=0}^{m-1} G^{(i)}(0) \left(\kappa_\varepsilon \phi_{m-\alpha} * \varphi_\varepsilon \right)^{(m-1-i)}.
\end{aligned}$$

Thus, we come up with the conclusion that, unless $G^{(i)}(0) = 0$, $i = 0, 1, \ldots, m-1$, fractional derivatives ${}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}(\tilde{G} * \varphi_\varepsilon)$ and $({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon}\tilde{G}) * \varphi_\varepsilon$ do not coincide!

Expectation. One of the questions one might be interested in is the quastion of expectation of the Caputo αth fractional derivative of the Colombeau generalized stochastic process $G(t) \in \mathcal{G}^{\Omega}((0,\infty))$. We consider it as follows.

For fixed $\varepsilon \in (0,1)$, suppose $E(G^{(k)}_\varepsilon(t)) = C_{k\varepsilon}(t)$, $\quad k \in \mathbb{N}$, $\quad t \geq 0$, where $G^{(k)}_\varepsilon(t)$ denotes the kth derivative of $G_\varepsilon(t)$ and E stands for the expectation. Then, for fixed $\varepsilon \in (0,1)$ and $m-1 < \alpha < m$, $m \in \mathbb{N}$, the following holds

$$E\left({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon} G_\varepsilon(t) \right) = \frac{1}{\Gamma(m-\alpha)} \int_0^{t-\varepsilon} \frac{C_{m\varepsilon}(\tau)}{(t-\tau)^{\alpha+1-m}}\, \mathrm{d}\tau.$$

The expectation $E\left({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon} G_\varepsilon(t) \right)$ exists, if the integral above converges, for example, when $C_{m\varepsilon}(\tau)$ is an absolutely continuous function. If $E(G^{(k)}_\varepsilon(t)) = C = \text{const}$, $t \geq 0$, then the following holds

$$E\left({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon} G_\varepsilon(t) \right) = \frac{C}{\Gamma(m-\alpha)}\, \frac{t^{m-\alpha} - \varepsilon^{m-\alpha}}{m-\alpha}, \;\; m-1 < \alpha < m,\; m \in \mathbb{N}.$$

For example, for Brownian motion viewed as a Colombeau generalized stochastic process, $W(t) \in \mathcal{G}^{\Omega}((0,\infty))$, we have $E\left({}^{\,c}_{\,0}D^{\alpha}_{t-\varepsilon} W_\varepsilon(t) \right) = 0$. This holds for all $\alpha > 0$ since the derivative of a Gaussian generalized stochastic process with mean zero is again Gaussian with mean zero.

5. RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES OF COLOMBEAU STOCHASTIC PROCESSES DEFINED ON $\mathbb{R}^+$

In this section we introduce the Riemann-Liouville αth fractional derivative of a Colombeau generalized stochastic process. We will see that the αth Riemann-Liouville fractional derivative of a Colomebau generalized stochastic process G defined on $(0,\infty)$ is a Colombeau generalized stochastic process itself, for every α, with no restrictions on G. On the other hand, if the Colombeau generalized stochastic process G is defined on $[0,\infty)$, the Riemann-Liouville fractional derivative of G is a Colombeau generalized stochastic process itself only if all derivatives of G_ε are vanishing at zero. But, as known from the classical fractional derivative theory, that is exactly the case when the Riemann-Liouville fractional derivative coincides with the Caputo fractional derivative.

Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$. The Riemann-Liouville αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, is defined by

$$(5) \qquad {}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_0^{t-\varepsilon} \dfrac{G_\varepsilon(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}, & m-1<\alpha<m \\ G_\varepsilon^{(m)}(t) = \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\, G_\varepsilon(t), & \alpha, = m \end{cases}$$

for $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$. As in the case of Caputo fractional derivative, for $m-1<\alpha<m$, $m \in \mathbb{N}$, changing the variables one obtains

$$\begin{aligned} {}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^{t-\varepsilon} \frac{G_\varepsilon(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_\varepsilon^{t} \frac{G_\varepsilon(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s. \end{aligned}$$

Lemma 3. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$. Let the Riemann-Liouville αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, ${}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t)$, $\alpha > 0$, be given by the relation* (5). *Then, for every $\alpha > 0$, we have $({}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t))_\varepsilon \in \mathcal{E}_M^\Omega((0,\infty))$.*

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. For $\alpha \in \mathbb{N}$, ${}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t)$ is the usual derivative of order α of $G_\varepsilon(t)$ and since $(G_\varepsilon(t))_\varepsilon \in \mathcal{E}_M^\Omega((0,\infty))$ the assertion immediately follows.

Now, consider the case when $m-1<\alpha<m$, $m \in \mathbb{N}$. First, note that

$$\begin{aligned} {}_0^{rl}D_{t-\varepsilon}^{\alpha} G_\varepsilon(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_\varepsilon^{t} \frac{G_\varepsilon(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\varepsilon^{t} \frac{G_\varepsilon(t-s)}{s^{\alpha+1-m}}\,\mathrm{d}s \right) \right] \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} \left[\int_\varepsilon^{t} \frac{G'_\varepsilon(t-s)}{s^{\alpha+1-m}}\mathrm{d}s + \frac{G_\varepsilon(0)}{t^{\alpha+1-m}} \right] \end{aligned}$$

$$= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^{m-2}}{\mathrm{d}t^{m-2}} \left[\int_\varepsilon^t \frac{G''_\varepsilon(t-s)}{s^{\alpha+1-m}} \mathrm{d}s + \frac{G'_\varepsilon(0)}{t^{\alpha+1-m}} + \frac{(m-\alpha-1)G_\varepsilon(0)}{t^{\alpha+2-m}} \right]$$

$$\vdots$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_\varepsilon^t \frac{G^{(m)}_\varepsilon(t-s)}{s^{\alpha+1-m}} \mathrm{d}s + \sum_{k=0}^{m-1} \frac{G^{(k)}_\varepsilon(0)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha},$$

i.e., the following holds:

$$(6) \qquad {}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t) = {}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t) + \sum_{k=0}^{m-1} \frac{G^{(k)}_\varepsilon(0)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha}.$$

Now we have

$$\sup_{t\in[\varepsilon,T]} |\,{}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)| \le \sup_{t\in[\varepsilon,T]} |\,{}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)| + \sum_{k=0}^{m-1} \sup_{t\in[\varepsilon,T]} \left| \frac{G^{(k)}_\varepsilon(0)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha} \right|.$$

But $k = 0, \dots m-1$ and $m-1 < \alpha < m$, $m \in \mathbb{N}$, providing that $k-\alpha$ is negative. Therefore,

$$\sup_{t\in[\varepsilon,T]} |\,{}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)| \le \sup_{t\in[\varepsilon,T]} |\,{}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)| + \sum_{k=0}^{m-1} \frac{G^{(k)}_\varepsilon(0)}{\Gamma(k+1-\alpha)}\, \varepsilon^{k-\alpha}.$$

We have proved in Lemma 1 that $\sup_{t\in[\varepsilon,T]} |\,{}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)|$ has a moderate bound, for every $\alpha > 0$. Therefore, $\sup_{t\in[\varepsilon,T]} |\,{}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)|$ has also a moderate bound, for every $\alpha > 0$.

To obtain a moderate bound for the first order derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right)$ we use the relation (6) and immediately obtain

$$\sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right) \right| \le \sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right) \right| + \sum_{k=0}^{m-1} \frac{G^{(k)}_\varepsilon(0)}{\Gamma(k-\alpha)} \varepsilon^{k-\alpha-1}.$$

But, as shown in Lemma 1, $\sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{c}_{0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right) \right|$ has a moderate bound and therefore $\sup_{t\in[\varepsilon,T]} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right) \right|$ has a moderate bound, too. Higher order derivatives can be estimated in a similar way. Thus, $({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_\varepsilon(t))_\varepsilon \in \mathcal{E}^\Omega_M((0,\infty))$, for every $\alpha > 0$.

Lemma 4. *Let $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ be two different representatives of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$. Then, for every $\alpha > 0$, we have $({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t))_\varepsilon - ({}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t))_\varepsilon \in \mathcal{N}^\Omega((0,\infty))$.*

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. For $\alpha \in \mathbb{N}$, ${}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_{1\varepsilon}(t)$ and ${}^{rl}_{\ 0}D^\alpha_{t-\varepsilon} G_{2\varepsilon}(t)$ are the usual derivatives of order α and since they represent the same Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega((0,\infty))$ the assertion immediately follows.

For $m-1<\alpha<m$, $m\in\mathbb{N}$, by using the relation (6), one obtains

$$\sup_{t\in[\varepsilon,T]} |\, {}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{1\varepsilon}(t) - {}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{2\varepsilon}(t)| \leq \sup_{t\in[\varepsilon,T]} |\, {}^{c}_{0}D^{\alpha}_{t-\varepsilon}G_{1\varepsilon}(t) - {}^{c}_{0}D^{\alpha}_{t-\varepsilon}G_{2\varepsilon}(t)| + \sum_{k=0}^{m-1} \frac{[G^{(k)}_{1\varepsilon}(0) - G^{(k)}_{2\varepsilon}(0))]}{\Gamma(k+1-\alpha)}\, \varepsilon^{k-\alpha}.$$

But, as proved in Lemma 2, $\sup_{t\in[\varepsilon,T]} |\, {}^{c}_{0}D^{\alpha}_{t-\varepsilon}G_{1\varepsilon}(t) - {}^{c}_{0}D^{\alpha}_{t-\varepsilon}G_{2\varepsilon}(t)|$ is negligible. Therefore, $\sup_{t\in[\varepsilon,T]} |\, {}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{1\varepsilon}(t) - {}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{2\varepsilon}(t)|$ is negligible, too. All derivatives can be estimated by using the similar argument. Thus, for every $\alpha>0$, $({}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{1\varepsilon}(t))_{\varepsilon} - ({}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{2\varepsilon}(t))_{\varepsilon} \in \mathcal{N}^{\Omega}((0,\infty))$. □

The Riemann-Liouville αth fractional derivative of a Colombeau generalized stochastic process on $(0,\infty)$ is now defined in the following definition.

Definition 3. *Let $G(t)\in\mathcal{G}^{\Omega}((0,\infty))$ be a Colombeau generalized stochastic process on $(0,\infty)$. The Riemann-Liouville αth fractional derivative of $G(t)$, in notation ${}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G(t) = \left[\left({}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{\varepsilon}(t)\right)_{\varepsilon}\right]$, $\alpha>0$, is an element of $\mathcal{G}^{\Omega}((0,\infty))$ satisfying (5).*

Remark 2. Note that one can similarly define the Riemann-Liouville αth fractional derivative of $G\in\mathcal{G}^{\Omega}([0,\infty))$, in notation ${}^{rl}_{0}D^{\alpha}_{t}G(t) = \left[\left({}^{rl}_{0}D^{\alpha}_{t}G_{\varepsilon}(t)\right)_{\varepsilon}\right]$, by

$${}^{rl}_{0}D^{\alpha}_{t}G_{\varepsilon}(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\displaystyle\int_0^t \frac{G_{\varepsilon}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}, & m-1<\alpha<m \\ G^{(m)}_{\varepsilon}(t) = \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\, G_{\varepsilon}(t), & \alpha=m \end{cases},$$

for $m\in\mathbb{N}$ and $\varepsilon\in(0,1)$. In that case one derives the same relation as (6), i.e.

$${}^{rl}_{0}D^{\alpha}_{t}G_{\varepsilon}(t) = {}^{c}_{0}D^{\alpha}_{t}G_{\varepsilon}(t) + \sum_{k=0}^{m-1}\frac{G^{(k)}_{\varepsilon}(0)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha},$$

for $m-1<\alpha<m$, $m\in\mathbb{N}$.

The problematic point is zero, and the derivative above is well defined in $t=0$ only if $G^{(k)}_{\varepsilon}(0)=0$, for $k=0,1,\dots,m-1$. But, in that case the Riemann-Liouville and the Caputo fractional derivatives coincide!

We conclude this section by briefly considering the expectation of the Riemann-Liouville αth fractional derivative of $G(t)\in\mathcal{G}^{\Omega}((0,\infty))$. As in the case of Caputo derivative, for fixed $\varepsilon\in(0,1)$, suppose $E(G^{(k)}_{\varepsilon}(t)) = C_{k\varepsilon}(t)$, $k\in\mathbb{N}$, $t\geq 0$, where $G^{(k)}_{\varepsilon}(t)$ denotes the kth derivative of $G_{\varepsilon}(t)$ and E stands for the expectation.

Then, for fixed $\varepsilon\in(0,1)$ and $m-1<\alpha<m$, $m\in\mathbb{N}$, the following holds

$$\begin{aligned} E\left({}^{rl}_{0}D^{\alpha}_{t-\varepsilon}G_{\varepsilon}(t)\right) &= E\left({}^{c}_{0}D^{\alpha}_{t-\varepsilon}G_{\varepsilon}(t)\right) + \sum_{k=0}^{m-1}\frac{E\left(G^{(k)}_{\varepsilon}(0)\right)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha} \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^{t-\varepsilon}\frac{C_{m\varepsilon}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau + \sum_{k=0}^{m-1}\frac{G_{k\varepsilon}}{\Gamma(k+1-\alpha)}\, t^{k-\alpha}, \end{aligned}$$

where we have denoted by $G_{k\varepsilon}$ the expectation of the random variable $G_\varepsilon^{(k)}(0)$, $k = 0, 1, \dots, m-1$. The expectation $E\left({}_0^{rl}D^\alpha_{t-\varepsilon} G_\varepsilon(t)\right)$ exists, if the integral and the sum above converge.

Cauchy problem with fractional derivatives

As an illustration of the theory, here we consider a Cauchy problem similar to the one in [**10**]:

$$X'(t) = a(t)\ X(t) + b(t)\ {}_0D^\alpha_{t-\varepsilon} G(t), \quad t \geq 0, \tag{7}$$

$$X(0) = X_0, \tag{8}$$

where ${}_0D^\alpha_{t-\varepsilon} G(t)$, $\alpha > 0$, can be both ${}_0^{c}D^\alpha_{t-\varepsilon} G(t)$ and ${}_0^{rl}D^\alpha_{t-\varepsilon} G(t)$, $G(t)$ is a Colombeau generalized stochastic process on $(0,\infty)$, i.e. $G(t) \in \mathcal{G}^\Omega((0,\infty))$ and $X_0 = [(X_{0\varepsilon})_\varepsilon]$ is a Colombeau generalized random variable.

We suppose that $a(t)$ is a deterministic, smooth function for $t \geq 0$ and denote $\bar{a}(\tau) = \int\limits_0^\tau a(t)\mathrm{d}t$. Function $b(t)$ is supposed to be deterministic and smooth for $t \geq 0$. We will show that under the conditions above, problem (7)–(8) has an almost surely unique solution $X(t) \in \mathcal{G}^\Omega((0,\infty))$. For that purpose, fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. The Cauchy problem (7)–(8) given by representatives reads

$$X'_\varepsilon(t) = a(t)\ X_\varepsilon(t) + b(t)\ {}_0D^\alpha_{t-\varepsilon} G_\varepsilon(t), \quad t \geq 0, \tag{9}$$

$$X_\varepsilon(0) = X_{0\varepsilon}, , \tag{10}$$

where $(G_\varepsilon)_\varepsilon \in \mathcal{E}^\Omega_M((0,\infty))$ and $(X_{0\varepsilon})_\varepsilon \in \mathcal{E}R_M$. The problem (9)–(10) has the solution

$$X_\varepsilon(t) = X_{0\varepsilon} e^{\bar{a}(t)} + e^{\bar{a}(t)} \int_0^t e^{-\bar{a}(\tau)}\ b(\tau)\ {}_0D^\alpha_{\tau-\varepsilon} G_\varepsilon(\tau)\ \mathrm{d}\tau. \tag{11}$$

To show that $(X_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}^\Omega_M((0,\infty))$ first note that from (11) we have

$$\begin{aligned} \sup_{t\in[0,T]} |X_\varepsilon(t)| &\leq |X_{0\varepsilon}|\ \exp\left(\sup_{t\in[0,T]} \bar{a}(t)\right) \\ &+ T \exp\left(\sup_{t\in[0,T]} \bar{a}(t) - \inf_{\tau\in[0,T]} \bar{a}(\tau)\right) \sup_{\tau\in[0,T]} |b(\tau)| \sup_{\tau\in[0,T]} |\ {}_0D^\alpha_{\tau-\varepsilon} G_\varepsilon(\tau)|. \end{aligned}$$

Since ${}_0D^\alpha_{t-\varepsilon} G(t) \in \left\{ {}_0^{c}D^\alpha_{t-\varepsilon} G(t),\ {}_0^{rl}D^\alpha_{t-\varepsilon} G(t)\right\}$, according to Lemma 1 and Lemma 3 we know that $\sup_{\tau\in[0,T]} |\ {}_0D^\alpha_{\tau-\varepsilon} G_\varepsilon(\tau)|$ has a moderate bound, for every $\alpha > 0$. Also, $(X_{0\varepsilon})_\varepsilon \in \mathcal{E}R_M$. Using that and the fact that $\bar{a}(t)$ and $b(t)$ are continuous, we obtain a moderate bound for $\sup_{t\in[0,T]} |X_\varepsilon(t)|$.

The moderate estimate for the first order derivative of X_ε follows directly from the equation (9). We obtain

$$\sup_{t\in[0,T]} |X'_\varepsilon(t)| \leq C_1 \sup_{t\in[0,T]} |X_\varepsilon(t)| + C_2 \sup_{t\in[0,T]} |\ {}_0D^\alpha_{t-\varepsilon} G_\varepsilon(t)|, \quad \text{for some } C_1, C_2 > 0.$$

Since both, $\sup_{t\in[0,T]} |\,{}_0D^{\alpha}_{t-\varepsilon}G_\varepsilon(t)|$ and $\sup_{t\in[0,T]} |X_\varepsilon(t)|$ have moderate bounds, we immediately conclude that $\sup_{t\in[0,T]} |X'_\varepsilon(t)|$ has a moderate bound, too.

By differentiating the equation (9) and by using the similar arguments as above, one obtains moderate bounds for higher order derivatives of X_ε. Thus, $(X_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}^{\Omega}_M((0,\infty))$.

One can easily show that the solution is almost surely unique in $\mathcal{G}^{\Omega}((0,\infty))$ by considering the equation

$$\bar{X}'_\varepsilon(t) = a(t)\,\bar{X}_\varepsilon(t) + N_\varepsilon(t), \quad \bar{X}_\varepsilon(0) = N_{0\varepsilon},$$

where $(\bar{X}_\varepsilon)_\varepsilon = (X_{1\varepsilon} - X_{2\varepsilon})_\varepsilon$ and $(X_{1\varepsilon})_\varepsilon, (X_{2\varepsilon})_\varepsilon \in \mathcal{E}^{\Omega}_M((0,\infty))$ are two solutions to the equation (9), $(N_\varepsilon)_\varepsilon \in \mathcal{N}^{\Omega}((0,\infty))$ and $(N_{0\varepsilon})_\varepsilon \in \mathcal{N}R$. After a similar procedure as in the existence part of the proof one obtains that $(X_{1\varepsilon} - X_{2\varepsilon})_\varepsilon \in \mathcal{N}^{\Omega}((0,\infty))$. Thus, the solution X to problem (7)–(8) is unique in $\mathcal{G}^{\Omega}((0,\infty))$.

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