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SINGULARITY ANALYSIS FOR AUTONOMOUS AND NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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Singularity analysis of ordinary differential equations is an important tool in the determination of the possible integrability of the equations. Although singularity analysis has been studied for decades, it still seems to cause problems in terms of the interpretation of some elements of the analysis. We summarise these problems and try to present a delicate approach towards their resolution by separating the treatment for autonomous and nonautonomous equations.

1. INTRODUCTION

In recent times singularity analysis has become an integral component of the standard approach to the analysis of differential equations, be they ordinary or partial, scalar or system. This development was a consequence of the observation that nonlinear partial differential equations solvable by means of the Inverse Scattering Method could be reduced to an equation of Painlevé Type [**1, 2, 3**]. The analysis was popularised in the works of Ramani et al. [**27**] and Tabor [**28**]. The development of singularity analysis can be found in the work of Kowalevskaya [**22**] in which she found the third integrable case of the top and her procedure forms the basis of what is now known as the Painlevé Test. About the same time Painlevé and his School [**24, 15, 16, 17**] began the long process of the classification of ordinary differential equations in terms of analytic functions. A concise discussion of these early results can be found in the classic text of Ince [**20**]. The process of classification proceeds steadily even to the present day as ordinary differential equations of higher order and partial differential equations are subjected to scrutiny. Some of this work can be found in references [**6, 7, 12, 19**]. We should emphasise that the process of classification and the application of the so-called Painlevé Test generally do not proceed by the same methods. In fact we are reminded of the comment of

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Painlevé that he did not see the need for 'le procédé connu de Madame Kowaleski' [**24**].

The Singularity Analysis in terms of its application to the resolution of a differential equation is essentially the determination of an expression for the solution of the equation as a Laurent expansion about a singularity which we take to be polelike. In the process the analysis can detect whether the singularity is a pole or a branch-point singularity. It cannot detect an essential singularity. If one follows the tradition of Kowalevskaya, clearly presented in the works of Ramani et al. and Tabor, the singularity analysis has a few simple criteria. The leading-order behaviour is a pole and the Kowalevskaya exponents, nowadays usually called the resonances, are nonnegative integers apart from a generic -1 in the standard analysis. It does happen that there exist equations with negative or mixed resonances, a fact that was indeed recognised as far as a century ago by Chazy [**7, 9, 21**] (see Section 4.1). The solution has a movable natural boundary – a closed curve in the complex plane (a circle in this case) beyond which the solution cannot be analytically continued [**9**]. If one thinks of this in terms of negative resonances, a solution starting from the neighbourhood of the singularity and expressed as a power series cannot pass beyond the circle which determines the lower value of the radius of convergence of the asymptotic expansion. The equation of Chazy, (3), seems to have remained an oddity without explanation. However, about twenty years ago various explanations for the case of negative resonances were put forward and for one interpretation the reader is referred to the perturbative approach by Conte, Fordy and Pickering [**10, 11, 14**].

There were also equations with a mixture of positive and negative resonances. Feix et al. [**13**] provided a rational explanation for negative resonances and the whole question – positive, negative, mixed resonances – was explained with explicit examples by Andriopoulos and Leach [**4**]. The explanation was quite simple; found within a first course in the theory of functions of a complex variable. A related matter, which had exercised the imagination of some workers in the field [**25**], was the interpretation of a singularity analysis which yielded an insufficient number of resonances to provide the number of arbitrary constants required for a general solution. An early example is found in Lemmer et al. [**23**] and a more comprehensive discussion is given by Rajasekar [**26**].

2. ASPECTS OF THE METHOD OF POLELIKE EXPANSIONS

We recall the standard algorithm of the Painlevé Test as applied to ordinary differential equations [**27, 28**]. Firstly one determines the leading-order behaviour by making the substitution

$$y_i = \alpha_i \chi^{p_i}, \quad i = 1, n, \tag{1}$$

into the given system of ordinary differential equations which have y_i, $i = 1, n$, dependent variables and $\chi = x - x_0$, where x_0 is the location of the putative polelike singularity. The location of the singularity is arbitrary in the absence of the application of initial/boundary conditions. The remaining constants of integration required to have a general solution of the system enter at the so-called resonances. After the leading-order behaviour has been established by the substitution of (1), one substitutes

$$y_i = \alpha_i \chi^{p_i} + \mu_i \chi^{p_i + r}, \quad i = 1, n, \tag{2}$$

into the system. The values of r, the resonances, are determined by requiring that terms linear in the μ_i have coefficient zero (A geometric interpretation using the same principles is found in [**18**]). Given that the coefficients of the μ_i are zero, the μ_i are arbitrary. Should there be a sufficient number of these arbitrary constants, there are enough constants to provide a general solution. Since the aim of the singularity analysis is to determine the integrability of the system in terms of analytic functions, it is evident that the leading-order exponents, p_i, $i = 1, n$, are negative integers and that the resonances take integral values. (The extension of this discussion to the case of the so-called 'weak Painlevé Property' is not particularly difficult)

After the leading-order behaviour has been determined one seeks the next-to-leading-order behaviour. Vital information is granted by the various values of the resonances which indicate the exact positions where the remaining arbitrary constants appear in the Laurent expansion representing the solution of the given equation. Goriely [**18**] provides a proof (following a different formulism) of the everoccurence of the -1 value. The other $n - 1$ resonances (for an nth-order differential equation) can in fact take any other value.

Every resonance is assigned a specific meaning – indeed the same for all possible values: the position where a constant of integration enters the expansion. What is in fact noteworthy is the meaning attributed to the value -1 throughout the years of the existance of the singularity analysis. Independently of the paper, every single publication treated the -1 resonance as an indication of the forcing of the singularity to be at the leading-order term or its occurence was associated with the freedom surrounding the putative singularity. Why? Is there a solid reason why one should interpret the value -1 differently from all other values?

We assume the standard $y = \alpha\chi^p + \mu\chi^{p+r}$ as the next-to-leading-order behaviour independently of the nature or the domain of convergence for the resulting series. The emergence of any value for the resonances signifies the insertion of an arbitrary constant at the $p + r$ power. Consider the case of -1. The above lead to the result that an arbitrary constant comes at $p-1$. True. In the case of nonpositive values for all resonances (see Sections 3.1 and 4.1 when dealing with Left Painlevé Series) there is no problem with -1 and the arbitrary constants are present at each power that the values of the resonances indicate. That the position of the putative singularity, x_0, is merged within those constants is a separate matter which is given its appropriate attention in what follows. On the other hand in the case of strictly nonnegative values for the other resonances one is confronted by a Right Painlevé Series. The Laurent expansion starts from the power p and then augments by increasing powers of χ, i.e., $p+1$, $p+2$, $\ldots$, where the coefficients of some terms may be zero. It therefore becomes apparent that the $p - 1$-term reflecting the -1 value for the resonance is more dominant than the actual dominant term, a fact which is indeed remarkable and cannot but be precluded; the resonance -1 is discarded on the grounds that it produces a term which is more dominant.

Observe that all the above come independently of the existance of a ∂_x symmetry and consequently the use of the argument that one may relocate the assumed pole from x_0 to 0. Whilst it is true that one could make the translation to the origin, the x_0 would have to come into the initial conditions. In the case of autonomous equations, the possession of the symmetry ∂_x is straightforward and one could have sought from the beginning for a series around zero and not around an arbitrary x_0 as both are equivalent under a translating transformation. This is quite different with what we wish to address here.

3. AUTONOMOUS EQUATIONS

3.1. An instance of a Left Painlevé Series

Consider the Chazy equation

$$y''' - 2yy'' + 3y'^2 = 0. \tag{3}$$

When we perform the standard singularity analysis, we find that $p = -1$, $\alpha_1 = -6$ and $r = -1, -2, -3$ (Note that the only subdominant possibility is $p = -2$ and α arbitrary). Most authors would therefore claim that, since all values of the resonances are nonpositive, the procedure ends and there is no representation in terms of a Laurent expansion. There are some others, on the other hand, who claim that a Series exists but in a domain removed from the singularity. There is also a third group that developed the perturbative approach with their interpretation of what one should do when confronted with negative resonances [**10, 11, 14**]. To make this clear we represent the Left Painlevé Series, as it is called, as

$$y(x) = -6\chi^{-1} + \alpha_2\chi^{-2} + \alpha_3\chi^{-3} + \alpha_4\chi^{-4} + \cdots, \tag{4}$$

where $\chi = x - x_0$. It is now clear that (4) contains (seemingly) four arbitrary constants, x_0 and α_i, $i = 2, 4$. This is explained in what follows.

After we have calculated the resonances, we are obliged to seek for consistency until the highest resonance. Of course in this case we have a series with decreasing powers and in fact we include a further term in order to illustrate a point. When we substitute the complete series into the whole equation, (3), we obtain (where there is summation on repeated indices)

$$i(i+1)(i+2)\alpha_i\chi^{-i-3} + (2i(i+1)\alpha_i\alpha_j - 3ij\alpha_i\alpha_j)\,\chi^{-i-j-2} = 0, \quad i, j = 1, 2, \ldots.$$

The χ^{-4} terms give $\alpha_1 = -6$, the terms χ^{-5}, χ^{-6} and χ^{-7} give the three arbitrary constants α_2, α_3 and α_4, respectively. If we collect the χ^{-8} terms, then we get that $\alpha_5 = (3\alpha_3^2 - 4\alpha_2\alpha_4)/6$ as expected since every further coefficient should be expressed in terms of the 'four' arbitrary constants. In order to show that this number is in fact three and considering the fact that the Series is valid outside a disc surrounding the singularity, *ie* $|x-x_0| > |\bar{x}-x_0|$, where $\bar{x}$ stands for all points on the boundary of the disc, we rewrite (4) as

$$\begin{aligned} y = &-\frac{6}{x}\frac{1}{\left(1-\frac{x_0}{x}\right)} + \frac{\alpha_2}{x^2}\frac{1}{\left(1-\frac{x_0}{x}\right)^2} + \frac{\alpha_3}{x^3}\frac{1}{\left(1-\frac{x_0}{x}\right)^3} + \frac{\alpha_4}{x^4}\frac{1}{\left(1-\frac{x_0}{x}\right)^4} \\ &+ \frac{\alpha_5}{x^5}\frac{1}{\left(1-\frac{x_0}{x}\right)^5} + \cdots = -\frac{6}{x}\left\{1+\frac{x_0}{x}+\left(\frac{x_0}{x}\right)^2+\left(\frac{x_0}{x}\right)^3+\left(\frac{x_0}{x}\right)^5+\cdots\right\} \\ &+ \frac{\alpha_2}{x^2}\left\{1+2\frac{x_0}{x}+3\left(\frac{x_0}{x}\right)^2+4\left(\frac{x_0}{x}\right)^3+\cdots\right\} \\ &+ \frac{\alpha_3}{x^3}\left\{1+3\frac{x_0}{x}+6\left(\frac{x_0}{x}\right)^2+\cdots\right\} + \frac{\alpha_4}{x^4}\left\{1+4\frac{x_0}{x}+\cdots\right\} + \frac{\alpha_5}{x^5}+\cdots \\ = &-\frac{6}{x} + \frac{1}{x^2}\left(-6x_0+\alpha_2\right) + \frac{1}{x^3}\left(-6x_0^2+2x_0\alpha_2+\alpha_3\right) \\ &+ \frac{1}{x^4}\left(-6x_0^3+3x_0^2\alpha_2+3x_0\alpha_3+\alpha_4\right) \\ &+ \frac{1}{x^5}\left(-6x_0^4+4x_0^3\alpha_2+6x_0^2\alpha_3+4x_0\alpha_4+\alpha_5\right)+\cdots. \end{aligned}$$

We now redefine the arbitrary constants as β_2, β_3 and β_4 by

$$\alpha_2 = \beta_2 + 6x_0, \quad \alpha_3 = \beta_3 - 2x_0\beta_2 - 6x_0^2, \quad \alpha_4 = \beta_4 - 3x_0\beta_3 + 3x_0^2\beta_2 + 6x_0^3$$

and, when all the above expressions are substituted into the coefficient of x^{-5}, we obtain $(3\beta_3^2 - 4\beta_2\beta_4)/6$. We conclude that the solution has 3 arbitrary constants.

Consider now the equation

$$y''' - 2yy'' + 3y'^2 + ky^8 = 0. \tag{5}$$

There are several points to be discussed in such a situation. We substitute for a leading-order term the usual $y = \alpha\chi^p$ in (5) to obtain

$$\alpha p(p-1)(p-2)\chi^{p-3} - 2\alpha^2 p(p-1)\chi^{2p-2} + 3\alpha^2 p^2\chi^{2p-2} + k\alpha^8\chi^{8p} = 0.$$

The only balance to be found (with an integral value) is $p=-1$, the last term is more dominant and in the current literature that value would not be permissible because all nondominant terms are supposed to be less dominant! However, when we look for a Left Painlevé Series, this is to be expected and the value $p=-1$ is accepted. We proceed for the resonances, which are determined by considering the dominant terms of the equation only. They are found to be $r=-1,\,-2,\,-3$ as before (this is how this example was constructed!). Indeed, when we substitute the truncated series up to the last resonance in all terms of (5), we recover the -6 value for the leading-order coefficient, the three arbitrary constants (which in fact absorb, as was explicitly demonstrated above, the position x_0 of the singularity) α_2, α_3 and α_4, and in the last step, when we collect all terms raised to the eightth power, we get

$$\alpha_5 = \frac{1}{2}{\alpha_3}^2 - \frac{2}{3}\alpha_2\alpha_4 - k{\alpha_1}^7,$$

which shows that there are no inconsistencies arising when all terms in (5) are taken into account. Therefore the series obtained is valid outside a punctured disc surrounding the pole.

3.2. An instance of a Right Painlevé Series

Consider another Chazy equation, *videlicet*

$$y''' - 2yy'' - 2y'^2 = 0. \tag{6}$$

The standard singularity analysis forces the leading-order exponent to be $p = -1$, the leading-order coefficient $\alpha_0 = -1$ and the resonances $r = -1, 2, 3$. We can therefore write the solution as the Laurent Series expansion

$$y(x) = -\chi^{-1} + \alpha\chi + \beta\chi^2 + \ldots, \tag{7}$$

which is valid in $|x - x_0| < |\bar{x} - x_0|$. If we require $|x| < |x_0|$, then (7) can be equally written as

$$\begin{aligned} y &= \frac{1}{x_0}\frac{1}{\left(1 - \dfrac{x}{x_0}\right)} + \alpha(x - x_0) + \beta(x - x_0)^2 + f_3(\alpha, \beta, x_0)(x - x_0)^3 + \cdots \\ &= \frac{1}{x_0}\left\{1 + \frac{x}{x_0} + \left(\frac{x}{x_0}\right)^2 + \ldots\right\} + \alpha(x - x_0) + \beta(x - x_0)^2 + \cdots \\ &= \left(\frac{1}{x_0} - \alpha x_0 + \beta x_0^2 + \sum_{i=3}^{\infty} f_i(\alpha, \beta, x_0)(-1)^i x_0^i\right) \\ &+ \left(\frac{1}{x_0^2} + \alpha - 2x_0\beta + \sum_{i=3}^{\infty} f_i(\alpha, \beta, x_0)(-1)^{i-1} i x_0^{i-1}\right) x \\ &+ \left(\frac{1}{x_0^3} + \beta + \sum_{i=3}^{\infty} f_i(\alpha, \beta, x_0)(-1)^{i-2}\frac{i(i-1)}{2}x_0^{i-2}\right) x^2 + \cdots \end{aligned}$$

and it becomes apparent that no merging of arbitrary constants can possibly occur.

4. NONAUTONOMOUS EQUATIONS

4.1. An instance of a Left Painlevé Series

Our vehicle is a modified Chazy equation, *videlicet*

$$y''' - 2yy'' + 3y'^2 + xy^9 = 0. \tag{8}$$

Apparently $p = -1$, $\alpha_1 = -6$ and $r = -1, -2, -3$. When the truncated series is substituted into the equation (where all terms are taken into account) and we collect all terms in χ^{-8}, we obtain $\alpha_5 = -\frac{1}{6}(4\alpha_2\alpha_4 - 3\alpha_3^2 - \alpha_1^9)$. It is at the χ^{-9} that the location of the pole enters explicitly into the expansion, i.e.

$$\alpha_6 = -\frac{1}{24}(-8\alpha_2^2\alpha_4 + 6\alpha_2\alpha_3^2 - 8\alpha_3\alpha_4 + 2\alpha_2\alpha_1^9 - 9\alpha_2\alpha_1^8 - x_0\alpha_1^9).$$

The Laurent expansion is therefore written as

$$y = -6\chi^{-1} + \alpha_2\chi^{-2} + \alpha_3\chi^{-3} + \alpha_4\chi^{-4} + \alpha_5\chi^{-5} + \alpha_6\chi^{-6},$$

where $\chi = x - x_0$, α_2, α_3 and α_4 are arbitrary and α_5 and α_6 are the expressions just above. Note that x_0 enters at the power $-9 - (-4) + (-1) = -6$ and that following the argument in a previous section the seemingly four arbitrary constants are essentially three.

4.2. An instance of a Right Painlevé Series

Consider the equation

$$y'' - 6y^2 - x = 0, \tag{9}$$

which is the first of the six Painlevé transcendents.

When we perform the singularity analysis, we obtain $p = -2$, $\alpha_0 = 1$ and $r = -1, 6$. The solution can therefore be written in the correct form as

$$y(x) = \chi^{-2} - \frac{x_0}{10}\chi^2 - \frac{1}{6}\chi^3 + \alpha_3\chi^4 + ..., \tag{10}$$

where $\chi = x - x_0$. Note that in (10) the x_0 enters the expansion at the fourth power after the leading order, or, otherwise put, at the power $0 - (-4) + (-2) = 2$.

Consider the equation

$$y'' - 2y^3 - xy - a = 0, \tag{11}$$

which is the second of the six Painlevé transcendents.

When we apply singularity analysis we obtain $p = -1$, $\alpha_0 = \pm 1$ and $r = -1, 4$. The solution can therefore be written in the correct form as

$$y(x) = \pm\chi^{-1} \mp \frac{x_0}{6}\chi + \frac{\mp 1 - a}{6}\chi^2 + \alpha_3\chi^3 + ..., \tag{12}$$

where $\chi = x - x_0$. This time the pole enters at the power $-1 - (-3) + (-1) = 1$.

In general (for nonautonomous equations) x_0 enters at the power given by the difference between the lowest power of χ multiplying x_0 in the equation and the dominant exponent in the equation plus the leading-order exponent. The arbitrary constants in each of (10) and (12) are two, x_0 and α_3, as it should be. Of course one notes the appearence of x_0 in the series. We claim that the pole appears in the coefficients of powers of χ in the solution of only nonautonomous equations. As has already been pointed out, the resonance -1 is discarded since it provides a term which is more dominant than the *a priori*-assumed leading-order term.

5. DISCUSSION/CONCLUSION

We have shown by way of several examples how one obtains any desired member of a Left or a Right Painlevé Series. These series are valid within particular domains of the complex plane. The question is now obvious: in the case of a full Laurent Series how does one determine the coefficient of any term in the Series expansion which is now valid within an annulus?

After the exponent(s) and coefficient(s) of the leading-order term(s) and the resonances have been calculated from the dominant terms and no problems have been encountered, the full Laurent series is substituted into the full equation to determine the coefficients of each term and to check for consistency at the resonances. Naturally, in terms of implementation, only a finite number of terms can ever be considered except for the (probably rare) cases in which it is possible to write an explicit expression for a recurrence relation defining the coefficients. For example, when we consider the third member of the Riccati differential sequence [**5**],

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0, \tag{13}$$

although all terms in (13) are dominant, the existence of two nongeneric resonances means that consistency needs to be checked. We have the three possibilities of a Left, a Right Painlevé Series and a full Laurent Series. For both the Left and the Right Painlevé Series, there is no inconsistency and all coefficients can be determined without ambiguity. In the case of the full Laurent Series the determination of coefficients poses an insurmountable difficulty. The coefficient of every exponent contains a doubly infinite number of terms and makes their determination in a precise form impossible. Even worse is the problem of consistency at the resonances. If one has to take into consideration the possibility of an infinite number of terms at each of the resonances, it is impossible to ensure consistency. It is possible that this difficulty led to the conclusion [**8**] that the introduction of a logarithmic term or terms was necessary for consistency. Of course the logarithmic terms destroy the analyticity of the solution. However, equation (13) has the explicit solution,

$$y = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3}, \tag{14}$$

where the constants, x_1, x_2 and x_3, are determined by the initial conditions and, without loss of generality, x_2 is closer to x_1 than x_3 and all of the expansions are about x_1. The Right Painlevé Series is defined in the region $0 < |x| < |x_2 - x_1|$. The Left Painlevé Series is valid for $|x| > |x_3 - x_1|$. The full series exists in the annulus defined by $|x_2 - x_1| < x < |x_3 - x_1|$. The expansions in the different regions of the series representations of the solution, (14), are completely devoid of logarithms. As the conclusion presented in Christiansen et al. [**8**] was a general statement, it is incumbent upon us to provide a rational explanation of the resolution of this problem. Fortunately equation (13) provides us with a very useful guide.

Our concern now is with differential equations which have both positive and negative resonances apart from the generic -1. Bearing in mind that the singularity analysis cannot reveal an essential singularity existence of such a series implies that it is valid on the exterior of a disc centred on the singularity. In the particular instance of (13) the region of validity is confined to an annulus, but one cannot exclude the possibility of validity over the whole of the complex plane exterior to the disc. Consequently there must be at least two singularities. The first is that about which the expansion is made formally and the second is the minimum value of $|x - x_2|$ for which the full series is convergent. In order to determine the coefficients of the full series all that one has to do is to move the point of expansion from x_1 to x_2. Then in a region about that point there exists a Right Painlevé Series and all of the coefficients can be determined. Naturally the region may not encompass the whole of the part of the complex plane encompassed by the full series. That is not a problem since the remaining region can be covered by means of analytic continuation. This does not mean that the process is simple. What it does mean is that it belongs to standard analysis. However, it is impossible to evaluate the coefficients in a finite algorithm.

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