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GROWTH OF MEROMORPHIC SOLUTIONS OF SOME DIFFERENCE EQUATIONS

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We investigate higher order difference equations and obtain some results on the growth of transcendental meromorphic solutions, which are complementary to the previous results. Examples are also given to show the sharpness of these results. We also investigate the growth of transcendental entire solutions of a homogeneous algebraic difference equation by using the difference analogue of Wiman-Valiron Theory.

1. INTRODUCTION AND RESULTS

Throughout the paper, we use standard notations in the Nevanlinna theory (see e.g. [**11**, **16**, **23**]). Let $f(z)$ be a meromorphic function. Here and in the following the word "meromorphic" means meromorphic in the whole complex plane. We also use notations $\rho(f)$ and $\mu(f)$ for the order and the lower order of $f(z)$ respectively. Moreover, we say that a meromorphic function g is small with respect to f if $T(r,g) = S(r,f)$, where $S(r,f) = o(T(r,f))$ outside a possible exceptional set of finite logarithmic measure.

Recently, there has been an increasing interest in studying complex differences and difference equations during the last decades, see for instance [**1-5**, **9-10**, **14-15**, **17**, **19**].

Laine, Rieppo and Silvennoinen [**17**] investigated several higher order difference equations. In particular, they obtained the following result.

Theorem A. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z+c_j)\right) = f(p(z)), \tag{1}$$

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where $\{J\}$ *is a collection of all non-empty subsets of* $\{1,2,\cdots,n\}$, c_j*'s are distinct complex constants and* $p(z)$ *is a polynomial of degree* $k \geq 2$. *Moreover, we assume that the coefficients* $\alpha_J(z)$ *are small functions relative to* f *and that* $n \geq k$. *Then*

$$T(r,f) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{\log n}{\log k}$.

A natural question can be posed here: What can be said about the case $k = 1$, which is not discussed in Theorem A. In the following, we consider the growth of meromorphic solutions of a difference equation more general than (1), when $k = 1$.

Theorem 1. *Suppose that* f *is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z+c_j) \right) = Q(z, f(p(z))), \tag{2}$$

where $\{J\}$ *is a collection of all non-empty subsets of* $\{1,2,\ldots,n\}$, c_j $(j = 1,\ldots,n)$ *are distinct complex constants,* $p(z) = az + b$, $a, b \in \mathbb{C}$ *and* $Q(z,u)$ *is a rational function in* u *of* $\deg_u Q = d$ (> 0). *We also suppose that all the coefficients of* (2) *are small functions relative to* f.

(i) *If* $0 < |a| < 1$, $d \geq n$, *we have*

$$\mu(f) \geq \frac{\log d - \log n}{-\log |a|}. \tag{3}$$

(ii) *If* $|a| > 1$, *we have* $d \leq n$ *and*

$$\rho(f) \leq \frac{\log n - \log d}{\log |a|}. \tag{4}$$

(iii) *If* $|a| = 1$, $d > n$, *then we have* $\rho(f) = \mu(f) = \infty$.

Theorem A can also be generalized into Theorem 2, concerning (2) instead of (1).

Theorem 2. *Suppose that* f *is a transcendental meromorphic solution of the equation* (2), *where* $\{J\}$, c_j*'s are the same as in Theorem* 1, $p(z) = p_k z^k + \cdots + p_1 z + p_0$ $(p_0, p_1, \ldots, p_k \in \mathbb{C})$ *of degree* $k \geq 2$, *and* $Q(z,u)$ *is a rational function in* u *of* $\deg_u Q = d$ (> 0). *We also suppose that all the coefficients of* (2) *are small functions relative to* f. *Then* $dk \leq n$, *and*

$$T(r,f) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{\log n - \log d}{\log k}$, $\varepsilon > 0$.

REMARK 1. Clearly, f satisfies $\mu(f) = \rho(f) = 0$.

The following examples show that both (3) and (4) in Theorem 1 are sharp, i.e. "$\geq$" and "$\leq$" in (3) and (4) respectively cannot be replaced by "$>$", "$<$" or "$=$".

EXAMPLE 1. A function $f(z) = e^z$ satisfies

$$f(z+c)f(z-c) + f(z+c) + f(z-c) = \left(f\left(\frac{1}{2}z\right)\right)^4 + (e^c + e^{-c})\left(f\left(\frac{1}{2}z\right)\right)^2,$$

where c is any nonzero complex constant. Clearly, we have

$$\mu(f) = \rho(f) = 1 = \frac{\log d - \log n}{-\log|a|},$$

where $n = 2 < 4 = d, a = \frac{1}{2} < 1$. This example shows that the equality in (3) can be arrived.

EXAMPLE 2. A function $f(z) = e^{z^2}$ satisfies

$$f(z+c) + f(z-c) = e^{c^2}\left(e^{2zc} + e^{-2zc}\right)\left(f\left(\frac{1}{2}z\right)\right)^4,$$

where c, n, d, a are the same as in Example 1. Moreover, we note that the coefficients e^{2zc} and e^{-2zc} are small functions relative to e^{z^2}. Clearly, we have

$$\mu(f) = \rho(f) = 2 > 1 = \frac{\log d - \log n}{-\log|a|},$$

which shows that the inequality in (3) may hold.

EXAMPLE 3. A function $f(z) = e^z$ satisfies

$$f(z+c)f(z-c)f(z+2c)f(z-2c) + f(z+c)f(z-c) = (f(2z))^2 + f(2z),$$

where c is any nonzero complex constant. Clearly, we have

$$\mu(f) = \rho(f) = 1 = \frac{\log n - \log d}{\log|a|},$$

where $n = 4 > 2 = d, a = 2 > 1$. This example shows that the equality in (4) can be arrived.

EXAMPLE 4. A function $f(z) = e^{z^2}$ satisfies

$$e^{-60c^2} f(z+c)f(z-c)f(z+2c)f(z-2c)f(z+3c)f(z-3c)f(z+4c)f(z-4c)$$
$$+e^{-(5/2)c^2} f(z+c)f(z-c)f\left(z+\frac{c}{2}\right)f\left(z-\frac{c}{2}\right) = (f(2z))^2 + f(2z),$$

where c, d, a are the same as in the above example, except for $n = 10$. Clearly, we have

$$\mu(f) = \rho(f) = 2 < \log_2 5 = \frac{\log n - \log d}{\log|a|},$$

which shows that the inequality in (4) may hold.

The following example illustrates the case (iii) of Theorem 1.

EXAMPLE 5. A function $f(z) = e^{e^z}$ satisfies

$$f(z+c_1) + f(z+c_2) + f(z+c_1)f(z+c_2) = f(z+b) + f(z+b)^2 + f(z+b)^3,$$

where $c_1 = \dfrac{c_2}{2} = b = \log 2$, $a = 1$, $d = 3 > 2 = n$. Clearly, $\rho(f) = \mu(f) = \infty$ holds, showing that Theorem 1 (iii) may hold.

In the following, we consider the growth of transcendental entire solutions of some kind of homogeneous algebraic difference equation.

We introduce some notations here. Let c be a fixed non-zero complex number, then the forward difference $\triangle_c^n f$ for each integer $n \in \mathbb{N}$ is defined in the standard way [**22**, pp. 52] by

$$\begin{aligned}
&\triangle_c^1 f(z) = \triangle_c f(z) = f(z+c) - f(z),\\
&\triangle_c^n f(z) = \triangle_c(\triangle_c^{n-1} f(z)) = \triangle_c^{n-1} f(z+c) - \triangle_c^{n-1} f(z), \qquad n \geq 2.
\end{aligned}$$

In particular, if $c = 1$, we use the usual difference notation $\triangle_c^n f(z) = \triangle^n f(z)$.

An algebraic difference polynomial is a finite sum of difference products, that is, an expression of the form

$$P(z,f) = \sum_{\lambda \in I} P_\lambda(z) f^{i_0} (\triangle f)^{i_1} \cdots (\triangle^n f)^{i_n},$$

where I is a finite set of multi-indices $\lambda = (i_0, i_1, \ldots, i_n)$, and $P_\lambda(z), \lambda \in I$ are meromorphic coefficients. We denote the degree and the weight of each monomial $P_\lambda(z) f^{i_0} (\triangle f)^{i_1} \cdots (\triangle^n f)^{i_n}$ of $P(z,f)$ by

$$d(\lambda) = i_0 + i_1 + \cdots + i_n \quad \text{and} \quad w(\lambda) = i_1 + 2i_2 + \cdots + n i_n$$

respectively. Then we denote the degree and the weight of $P(z,f)$ by

$$d(P) = \max_{\lambda \in I}\{d(\lambda)\} \quad \text{and} \quad w(P) = \max_{\lambda \in I}\{w(\lambda)\}$$

respectively. In particular, if each monomial of $P(z,f)$ is of the same degree, we call $P(z,f)$ is a homogeneous algebraic difference polynomial, and $P(z,f) = 0$ is a homogeneous algebraic difference equation.

Recently, CHIANG and FENG [**5**] investigated point-wise estimates for difference quotient and applied to difference equations, obtaining the following theorem.

Theorem B. *Let $Q_0(z), Q_1(z), \ldots, Q_n(z)$ be polynomials such that there exists an integer $\ell, 0 \leq \ell \leq n$ so that*

$$\deg(Q_\ell) > \max_{\substack{0 \leq j \leq n \\ j \neq \ell}} \{\deg(Q_j)\}$$

holds. Suppose that $f(z)$ is a meromorphic solution to the difference equation

$$Q_n(z)f(z+n) + \cdots + Q_1(z)f(z+1) + Q_0(z)f(z) = 0, \tag{5}$$

then we have $\rho(f) \geq 1$.

Chiang and Feng [**4**] also applied higher order difference quotient estimates to difference equations and obtained the following theorem, which generalizes Theorem B.

Theorem C. Let $P_0(z), P_1(z), \ldots, P_n(z)$ be polynomials such that

$$\deg(P_0) \geq \max_{1\leq \ell \leq n}\{\deg(P_\ell)\}.$$

Suppose that $f(z)$ is a meromorphic solution to the difference equation

$$P_n(z)\triangle^n f(z) + \cdots + P_1(z)\triangle f(z) + P_0(z)f(z) = 0, \tag{6}$$

then we have $\rho(f) \geq 1$.

Remark 2. Note that equations (5) and (6) are both linear difference equations.

The following theorem is also from [**4**].

Theorem D. *Suppose that f is a transcendental entire solution of the first order algebraic difference equation*

$$P(z, f(z), \triangle f(z)) = 0, \tag{7}$$

with polynomial coefficients, then $\rho(f) > 0$.

In what follows, we consider a homogenous algebraic difference equation, which is more general than (5)-(7), by using the difference analogue of Wiman-Valiron Theory in the entire solution case, and obtain the following results, which generalizes Theorems C and D to some extent.

Theorem 3. *Suppose that f is a transcendental entire solution of a homogeneous algebraic difference equation of the form*

$$P(z,f) = \sum_{\lambda\in I} P_\lambda(z) f^{i_0}(\triangle f)^{i_1}\cdots(\triangle^n f)^{i_n} = 0, \tag{8}$$

with polynomial coefficients $P_\lambda(z)$, $\lambda \in I$, where I is a finite set of multi-indices $\lambda = (i_0, i_1, \ldots, i_n)$. If f satisfies $\rho(f) < 1$, then we have that

$$\rho(f) = \chi,$$

where χ is a rational number which can be determined from a gradient of the corresponding Newton-Puisseux diagram for the linear differential equation

$$\sum_{\lambda\in I} P_\lambda(z) g^{(w(\lambda))} = 0,$$

where $w(\lambda) = i_1 + 2i_2 + \cdots + ni_n, \lambda \in I$. *In particular,* $\chi > 0$.

The following theorem is closely related in spirit to Theorem C in [**13**].

Theorem 4. *Suppose that* f *is a transcendental entire solution of a homogeneous algebraic difference equation of the form* (8), *with polynomial coefficients* $P_\lambda(z),\ \lambda \in I$, *where* I *is a finite set of multi-indices* $\lambda = (i_0, i_1, \cdots, i_n)$. *If* w *is a positive integer such that*

$$w(\lambda) = i_1 + 2i_2 + \cdots + ni_n = w \tag{9}$$

holds for all $\lambda \in I$, *and*

$$\sum_{\lambda \in I} P_\lambda(z) \not\equiv 0, \tag{10}$$

then $\rho(f) \geq 1$.

The following example illustrates Theorem 4.

Example 6. A function $f(z) = ze^z$ satisfies the homogeneous algebraic difference equation

$$P_1(z)f^4 \triangle f \triangle^2 f \triangle^3 f + P_2(z) f(\triangle f)^6 = 0,$$

where $P_1(z) = -((e-1)z+e)^5$, $P_2(z) = z^3\big((e-1)^2 z + 2e^2 - 2e\big)\big((e-1)^3 z + 3e^3 - 6e^2 + 3e\big)$. Clearly, the assumptions (9) and (10) are satisfied and we have $\rho(f) = 1$, showing that Theorem 4 may hold.

2. AUXILIARY RESULTS

The following Lemma 1, due to Valiron [**21**] and Mohon'ko [**18**], has proved to be an extremely useful tool in the study of meromorphic solutions of differential, difference and functional equations.

Lemma 1. [**16, 18, 21**] (Valiron-Mohon'ko). *Let* $f(z)$ *be a meromorphic function. Then, for all irreducible rational functions in* f,

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, *the characteristic function of* $R(z, f(z))$ *satisfies*

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ *and* $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2. **[17]** *Given distinct complex numbers* $c_1, \ldots, c_n$*, a meromorphic function* f*, and small functions* $\alpha_J(z)$ *relative to* f*, we have*

$$T\left(r, \sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z+c_j)\right)\right) \le \sum_{k=1}^{n} T(r, f(z+c_k)) + S(r,f),$$

where $\{J\}$ *is a collection of all non-empty subsets of* $\{1, 2, \ldots, n\}$.

Lemma 3. **[1]** *Given* $\varepsilon > 0$ *and a meromorphic function* f*, the Nevanlinna characteristic function* T *satisfies*

$$T(r, f(z\pm 1)) \le (1+\varepsilon)T(r+1, f) + K,$$

for all $r \ge \dfrac{1}{\varepsilon}$ *and some constant* K*. Similarly, we have that*

$$T(r, f(z\pm c)) \le (1+\varepsilon)T(r+|c|, f)$$

holds for $\varepsilon > 0$ *and all* $r > r_0$*, where* r_0 *is some positive constant.*

Lemma 4. **[7]** *Let* $f(z)$ *be a transcendental meromorphic function and* $p(z) = p_k z^k + p_{k-1}z^{k-1} + \ldots + p_1 z + p_0$ *be a complex polynomial of degree* $k > 0$*. For given* $0 < \delta < |p_k|$*, let* $\lambda = |p_k| + \delta, \mu = |p_k| - \delta$*, then for given* $\varepsilon > 0$ *and for* r *large enough,*

$$(1-\varepsilon)T(\mu r^k, f) \le T(r, f\circ p) \le (1+\varepsilon)T(\lambda r^k, f).$$

Lemma 5. **[8]** *Let* $g : (0, +\infty) \to R,\ h : (0, +\infty) \to R$ *be nondecreasing functions. If* (i) $g(r) \le h(r)$ *outside of an exceptional set of finite linear measure, or* (ii) $g(r) \le h(r), r \notin H \cup (0, 1]$*, where* $H \subset (1, \infty)$ *is a set of finite logarithmic measure, then for any* $\alpha > 1$*, there exists* $r_0 > 0$ *such that* $g(r) \le h(\alpha r)$ *for all* $r > r_0$.

Lemma 6. **[19]** *Let* $\phi : (r_0, \infty) \to (1, \infty)$*, where* $r_0 \ge 1$*, be a monotone increasing function. If for some real constant* $\alpha > 1$*, there exists a real number* $K > 1$ *such that* $\phi(\alpha r) \ge K\phi(r)$*, then*

$$\varliminf_{r\to\infty} \frac{\log \phi(r)}{\log r} \ge \frac{\log K}{\log \alpha}\,.$$

Lemma 7. **[6]** *Let* $\psi(r)$ *be a function of* $r (r \ge r_0)$*, positive and bounded in every finite interval.*

(i) *Suppose that* $\psi(\mu r^m) \le A\psi(r) + B$ $(r \ge r_0)$*, where* $\mu(\mu > 0)$*,* $m(m > 1)$*,* $A(A \ge 1)$*,* B *are constants. Then* $\psi(r) = O((\log r)^\alpha)$ *with* $\alpha = \dfrac{\log A}{\log m}$*, unless* $A = 1$ *and* $B > 0$*; and if* $A = 1$ *and* $B > 0$*, then for any* $\varepsilon > 0$*,* $\psi(r) = O((\log r)^\varepsilon)$.

(ii) *Suppose that* (*with the notation of* (i)) $\psi(\mu r^m) \ge A\psi(r)$ $(r \ge r_0)$*. Then for all sufficiently large values of* r*,* $\psi(r) \ge K(\log r)^\alpha$ *with* $\alpha = \dfrac{\log A}{\log m}$*, for some positive constant* K.

Lemma 8. [**2**] *Let $n \in \mathbb{N}$ and f be transcendental and meromorphic of order less than 1 in the plane. Then there exists an ε-set E_n such that $\triangle^n f(z) \sim f^{(n)}(z)$, as $z \to \infty$ in $\mathbb{C}\backslash E_n$, i.e. $\dfrac{\triangle^n f(z)}{f^{(n)}(z)} \to 1$ as $z \to \infty$ in $\mathbb{C}\backslash E_n$.*

Lemma 9. [**16**] *Let f be a transcendental entire function, let $0 < \delta < \dfrac{1}{4}$ and z be such that $|z| = r$ and that*

$$|f(z)| > M(r,f)\nu(r,f)^{-1/4+\delta}$$

holds, where $\nu(r,f)$ is the central index of f. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure such that

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1))$$

holds for all $k \in \mathbb{N}$, $r \notin F$.

Lemma 10. [**16**] *If f is a non-constant entire function of order ρ, then*

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log \nu(r,f)}{\log r},$$

where $\nu(r,f)$ is the cental index of f.

Remark 3. [**12**] Following Hayman [**12**, pp. 75-76], we define an ε-set E to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure and hence zero logarithmic density, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

3. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. (i) We may suppose that $d > n$, since the case $d = n$ is trivial.

Suppose that $d_1(z), d_2(z), \ldots, d_\nu(z)$ are all coefficients of $Q(z, f(p(z)))$. Denote $\Psi(r) = \max\{T(r,d_i) | i = 1, 2, \ldots, \nu\}$, and $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. Applying Lemma 1 to the right-hand side of (2), we obtain by Lemmas 2 and 3 that

$$\begin{aligned} (11)\qquad & dT(r, f(p(z))) + O(\Psi(r)) = T(r, Q(z, f(p(z)))) \\ & = T\left(r, \sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z+c_j)\right)\right) \leq \sum_{k=1}^n T(r, f(z+c_k)) + S(r,f) \\ & \leq \left(1 + \frac{\varepsilon}{2}\right) \sum_{k=1}^n T(r+C, f) + S(r,f) \leq n\left(1+\frac{\varepsilon}{2}\right) T(\beta r, f) + S(r,f), \end{aligned}$$

for sufficiently large r and any given $\beta > 1$, $\varepsilon > 0$. By Lemma 4 and (11), we obtain that for $\mu = |a| - \delta(0 < \delta < |a|, 0 < \mu < 1)$ and sufficiently large r,

$$d(1-\varepsilon)T(\mu r, f) \le n(1+\varepsilon)T(\beta r, f)$$

outside of a possible exceptional set of finite linear measure. By Lemma 5, we obtain that for any given $\gamma > 1$ and sufficiently large r,

$$d(1-\varepsilon)T(\mu r, f) \le n(1+\varepsilon)T(\beta\gamma r, f),$$

that is

$$\frac{d(1-\varepsilon)}{n(1+\varepsilon)}T(r,f) \le T\left(\frac{\beta\gamma}{\mu}r, f\right).$$

Since $\beta, \gamma > 1$, $\mu < 1$ and $d > n$, we note that $\frac{\beta\gamma}{\mu} > 1$ and $\frac{d(1-\varepsilon)}{n(1+\varepsilon)} > 1$ for sufficiently small ε. Then by Lemma 6, we obtain that

$$\mu(f) \ge \frac{\log d(1-\varepsilon) - \log n(1+\varepsilon)}{\log \beta\gamma - \log \mu}.$$

Letting $\varepsilon \to 0$, $\delta \to 0$, and $\beta, \gamma \to 1$, we obtain that

$$\mu(f) \ge \frac{\log d - \log n}{-\log |a|}.$$

(ii) Similarly as in (i), we obtain that

$$d(1-\varepsilon)T(\mu r, f) \le dT(r, f(p(z))) \le n\left(1 + \frac{\varepsilon}{2}\right)T(r+C, f) + S(r,f),$$

where $\mu = |a| - \delta$ ($\delta > 0$ is chosen to be such that $\mu > 1$), and r is sufficiently large. We can select sufficiently small $\varepsilon > 0$ such that $\frac{1}{\mu} + \varepsilon < 1$. Therefore, we have

$$T(r,f) \le \frac{n(1+\varepsilon)}{d(1-\varepsilon)}T\left(\frac{r+C}{\mu}, f\right) \le \frac{n(1+\varepsilon)}{d(1-\varepsilon)}T\left(\left(\frac{1}{\mu}+\varepsilon\right)r, f\right)$$

outside of a possible exceptional set of finite logarithmic measure. We immediately note that $d \le n$. Applying Lemma 3.1 in [**9**], where we use Lemma 4 to get rid of the possible exceptional set of finite logarithmic measure, we obtain that

$$\rho(f) \le \frac{\log n(1+\varepsilon) - \log d(1-\varepsilon)}{-\log\left(\frac{1}{\mu}+\varepsilon\right)}.$$

Letting $\varepsilon \to 0, \delta \to 0$, we obtain that

$$\rho(f) \le \frac{\log n - \log d}{\log |a|}.$$

(iii) The case $|a| = 1,\ d > n$ follows the case $0 < |a| < 1,\ d > n$ in (i), word by word. In fact, when $|a| = 1,\ d > n$, we can also set $\mu = |a| - \delta = 1 - \delta$ $(0 < \delta < 1,\ 0 < \mu < 1)$, similarly as in (i). Then we have

$$\mu(f) \geq \frac{\log d - \log n}{-\log |a|}.$$

Since $|a| = 1$, we obtain that $\rho(f) = \mu(f) = \infty$. □

Proof of Theorem 2. By Lemmas 1-3, we obtain (11) here, where $\Psi(r)$, C, β and ε are defined as in the proof of Theorem 1. By Lemma 4, we obtain that for $\mu = |p_k| - \delta(> 0)$ and sufficiently large r,

$$d(1-\varepsilon)T(\mu r^k, f) \leq n(1+\varepsilon)T(\beta r, f)$$

outside of a possible exceptional set of finite linear measure. By Lemma 5, we obtain that for any given $\gamma > 1$ and sufficiently large r,

$$d(1-\varepsilon)T(\mu r^k, f) \leq n(1+\varepsilon)T(\beta\gamma r, f),$$

that is

$$T\left(\frac{\mu}{(\beta\gamma)^k}t^k, f\right) \leq \frac{n(1+\varepsilon)}{d(1-\varepsilon)}T(t, f),$$

where $t = \beta\gamma r$. We immediately note that $d \leq n$, since $k \geq 2$. It follows by Lemma 7 that

$$T(r, f) = O((\log r)^{\alpha_1}),$$

where $\alpha_1 = \dfrac{\log n(1+\varepsilon) - \log d(1-\varepsilon)}{\log k} = \dfrac{\log n - \log d}{\log k} + \varepsilon_1$. Denoting $\alpha = \dfrac{\log n - \log d}{\log k}$, we have that

$$T(r, f) = O((\log r)^{\alpha+\varepsilon_1}).$$

Finally, we affirm that $dk \leq n$. Otherwise, the contrary fact that $dk > n$ results in $\alpha < 1$, then we have $\alpha + \varepsilon_1 = \alpha_1 < 1$ for sufficiently small $\varepsilon_1 > 0$, which contradicts with the transcendency of f. □

4. PROOFS OF THEOREMS 3 AND 4

Proof of Theorem 3. Since the equation (8) is homogeneous, say of degree d, we may divide it by f^d and have that

$$\sum_{\lambda \in I} P_\lambda(z)\left(\frac{\triangle f}{f}\right)^{i_1} \cdots \left(\frac{\triangle^n f}{f}\right)^{i_n} = 0. \tag{12}$$

By Lemma 8 and the assumption that $\rho(f) < 1$, there exists an ε-set E such that as $z \to \infty,\ z \in \mathbb{C}\backslash E$,

$$\frac{\triangle^k f(z)}{f(z)} \sim \frac{f^{(k)}(z)}{f(z)},\ k = 1, \ldots, n. \tag{13}$$

Denoting

$$F_1 = \{r = |z| \in (1, \infty) : z \in E\},$$

we have by REMARK 3 that F_1 is of finite logarithmic measure. By Lemma 9, we have that

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1)),\ k = 1, \dots, n, \tag{14}$$

for all large z satisfying $|z| = r \notin F_2$ and $|f(z)| = M(r, f)$, where $F_2 \subset \mathbb{R}_+$ is of finite logarithmic measure and $\nu(r, f)$ is the central index of f. By (13) and (14), we have that

$$\frac{\triangle^k f(z)}{f(z)} = \frac{f^{(k)}(z)}{f(z)}(1 + o(1)) = \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1)),\ k = 1, \dots, n, \tag{15}$$

for all large z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $|f(z)| = M(r, f)$. Applying (15) to (12), we have that

$$\sum_{\lambda \in I} A_\lambda z^{p_\lambda} \left(\frac{\nu(r,f)}{z}\right)^{w(\lambda)} (1 + o(1)) = 0, \tag{16}$$

where A_λ are the leading coefficients of $P_\lambda(z)$, p_λ are the degrees of $P_\lambda(z)$, and $w(\lambda) = i_1 + 2i_2 + \cdots + ni_n$ respectively. Since solutions of an algebraic equation are continuous functions of coefficients, $\nu(r, f)$ approximately equals to the solution of

$$\sum_{\lambda \in I} A_\lambda z^{p_\lambda - w(\lambda)} \nu(r, f)^{w(\lambda)} = 0. \tag{17}$$

By Valiron [**20**, p.108 and Appendix A], we have that

$$\nu(r, f) \sim B r^\chi, \tag{18}$$

for all sufficiently large $r \notin [0, 1] \cup F_1 \cup F_2$, where B is a constant and χ is a fixed rational number which can only take values equal to the gradients of the corresponding Newton-Puisseux diagram for the linear differential equation

$$\sum_{\lambda \in I} P_\lambda(z) g^{(w(\lambda))} = 0. \tag{19}$$

In fact, assuming that g is a transcendental entire solution of (19), we may apply Lemma 9 to (19) and have that $\nu(r, g)$ satisfies

$$\sum_{\lambda \in I} A_\lambda z^{p_\lambda} \left(\frac{\nu(r,g)}{z}\right)^{w(\lambda)} (1 + o(1)) = 0,\ r \notin F_3, \tag{20}$$

where $F_3 \subset \mathbb{R}_+$ is of finite logarithmic measure and $\nu(r, g)$ is the central index of g. Similar to (16), (20) means that $\nu(r, g)$ also approximately equals to the solution of (17). By Valiron [**20**, p.108 and Appendix A] again, we have that

$$\nu(r, g) \sim B_1 r^{\chi_1},\ r \notin F_3,$$

where B_1 is a constant and χ_1 is a fixed rational number which can only take values equal to the gradients of the corresponding Newton-Puisseux diagram for (19). Thus, the possible values of χ are the same as χ_1. Finally, by (18) and Lemmas 5 and 10, we have that $\rho(f) = \chi$. In particular, $\chi > 0$. □

Proof of Theorem 4. Assume to the contrary that $\rho(f) < 1$. Since the equation (8) is homogeneous, say of degree d, we may rewrite it in the form

$$\sum_{\lambda \in I} P_\lambda(z) \frac{f^{i_0}(\triangle f)^{i_1} \cdots (\triangle^n f)^{i_n}}{f^d} = 0. \tag{21}$$

Since $\rho(f) < 1$, we have that (13)-(15) hold as in the proof of Theorem 3. Applying (15) to (21), we have by (9) that

$$\left(\sum_{\lambda \in I} P_\lambda(z) \right) \left(\frac{\nu(r,f)}{z} \right)^w (1 + o(1)) = 0, \tag{22}$$

where $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and z is chosen such that $|f(z)| = M(r,f)$, and where $F_1,\ F_2$ are the same as in the proof of Theorem 3. It follows by (10) and (22) that $\nu(r,f) \equiv 0$, which contradicts with the transcendency of f. Hence, $\rho(f) \geq 1$.□

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