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SOME RESULTS ON MATCHING AND TOTAL DOMINATION IN GRAPHS

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Let G be a graph. A set S of vertices of G is called a total dominating set of G if every vertex of G is adjacent to at least one vertex in S. The total domination number $\gamma_t(G)$ and the matching number $\alpha'(G)$ of G are the cardinalities of the minimum total dominating set and the maximum matching of G, respectively. In this paper, we introduce an upper bound of the difference between $\gamma_t(G)$ and $\alpha'(G)$. We also characterize every tree T with $\gamma_t(T) \leq \alpha'(T)$, and give a family of graphs with $\gamma_t(G) \leq \alpha'(G)$.

1. INTRODUCTION

Domination and its variants in graphs have been being well-studied in the past decade. The literature on this subject has been surveyed thoroughly in the two books by Haynes, Hedetniemi and Slater [**4, 5**].

Let $G = (V, E)$ be a simple graph of order n. A *matching* M in a graph G is a set of independent edges in G. A vertex v of G is *saturated by* M if it is the endpoint of an edge of M; otherwise, vertex v is *unsaturated by* M. An *induced matching* M is a matching where no two edges of M are joined by an edge of G. The *matching number* $\alpha'(G)$ and the *induced matching number* $im(G)$ are the cardinalities of a maximum matching and a maximum induced matching of G, respectively. It is obvious that any induced matching is a matching. So $\alpha'(G) \geq im(G)$.

Let V be the set of vertices of G. A set S $(\subseteq V)$ is called a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S. A total

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dominating set, which was introduced by Cockayne, Dawes, and Hedetniemi [**2**], is a dominating set of G containing no isolated vertices. The *total domination number* $\gamma_t(G)$ of G is the cardinality of a minimum total dominating set.

We in general follow the notation and graph terminology in [**4, 5**]. Specifically, the degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. For a subset S of V, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. For disjoint subsets S_1 and S_2 of V, we define $G[S_1, S_2]$ as the set of edges of G joining S_1 and S_2. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. A vertex of degree one is called a *leaf.* A vertex v of G is called a *support* if it is adjacent to a leaf. Let $L(G)$ and $S(G)$ denote the set of leaves and supports of G, respectively. A *star* is the tree $K_{1,n-1}$ of order $n \geq 2$.

Henning et al. [**8**] investigated the relationships between the matching and total domination number of a graph. They showed that the matching number and the total domination number of a graph are incomparable, even for an arbitrarily large, but fixed (with respect to the order of the graph), minimum degree.

Theorem 1.1. ([**8**]) *For every integer $\delta \geq 2$, there exist graphs G and H with $\delta(G) = \delta(H) = \delta$ satisfying $\gamma_t(G) > \alpha'(G)$ and $\gamma_t(H) < \alpha'(H)$.*

It is obvious that $\gamma_t(G) \leq 2\alpha'(G)$. That is,

$$\gamma_t(G) - \alpha'(G) \leq \alpha'(G) \leq \frac{n}{2}.$$

We shall improve this bound in Section 2. First, we prove that, for any connected graph G,

$$\gamma_t(G) - \left(\frac{\delta(\Delta+\delta)+\delta-1}{\delta(\Delta+\delta)}\right) im(G) \leq \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right),$$

and characterize the extremal graphs. Then, we work out an upper bound on the difference between the total domination number and the matching number.

Theorem 1.2. ([**8**]) *For every claw-free graph G with $\delta(G) \geq 3$, $\gamma_t(G) \leq \alpha'(G)$.*

Theorem 1.3. ([**8**]) *For every k-regular graph G with $k \geq 3$, $\gamma_t(G) \leq \alpha'(G)$.*

Furthermore, Henning et al. raised the following question: Find other families of graphs with total domination number at most their matching number.

Recently, Henning and Yeo [**7**] characterized the connected claw-free graphs with minimum degree at least three that have equal total domination and matching number.

In this paper, we obtain an upper bound on the difference between the total domination number and the matching number in Section 2. In Section 3, we characterize all trees and give a family of graphs with the total domination numbers at most their matching numbers.

2. AN UPPER BOUND ON THE DIFFERENCE BETWEEN THE TOTAL DOMINATION NUMBER AND THE MATCHING NUMBER

In this section, we present an upper bound on the difference between the total domination number and the matching number in terms of the minimum degree, maximum degree, order and induced matching number.

Lemma 2.1. ([**6**]) *Let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of degree at least $\delta \geq 1$. Then there exists a set $A \subseteq X$ of cardinality at most $\frac{1}{2}\Big(|Y| + \frac{|X|}{\delta}\Big)$ dominating Y.*

Theorem 2.2. *For any connected graph G,*

$$\gamma_t(G) - \left(\frac{\delta(\Delta+\delta)+\delta-1}{\delta(\Delta+\delta)}\right) im(G) \leq \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right).$$

Furthermore the equality holds if and only if G is isomorphic to either P_2 or C_5.

Proof. Let M be a maximum induced matching of G and let S_1 be the set of saturated vertices by M. Define $S_2 = N(S_1) - S_1$ and $S_3 = V - N[S_1]$. It is obvious that $|S_1| + |S_2| + |S_3| = n$.

Suppose $S_3 = \varnothing$. Then $\gamma_t(G) \leq |S_1| = 2im(G)$ and $im(G) \leq \frac{\Delta}{\Delta+\delta-1}\,\frac{n}{2}$. Hence

$$\begin{aligned}\gamma_t(G) - \left(\frac{\delta(\Delta+\delta)+\delta-1}{\delta(\Delta+\delta)}\right) im(G) &\leq \left(\frac{\delta(\Delta+\delta)-\delta+1}{\delta(\Delta+\delta)}\right) im(G)\\ &= \left(\frac{(\Delta+\delta)-1+1/\delta}{(\Delta+\delta)}\right) im(G)\\ &\leq \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right)\end{aligned}$$

If the equality holds, then $im(G) = \frac{n}{2}$ and $\delta = 1$. Hence $G \cong P_2$.

Suppose $S_3 \neq \varnothing$ and there is an edge $e = uv \in E(G[S_3])$. Since both u and v are at distance at least 2 from S_1, it follows that $M \cup \{e\}$ is an induced matching of G larger than M, which is a contradiction. Therefore, S_3 is an independent set.

Let H be the bipartite subgraph of G with partite sets $(S_3, N(S_3))$ and with the edge set defined by $G[S_3, N(S_3)]$. Then each vertex in S_3 is of degree at least $\delta \geq 1$ in H. By Lemma 2.1, there exists a set $A \subseteq N(S_3)$ of cardinality at most $\frac{1}{2}\Big(|S_3| + \frac{|N(S_3)|}{\delta}\Big)$ dominating S_3. Since $N(S_3) \subseteq S_2$, it follows that

$$|A| \leq \frac{1}{2}\Big(|S_3| + \frac{|S_2|}{\delta}\Big).$$

As the number of edges joining $S_1 \cup S_3$ and S_2 satisfies $(\delta-1)|S_1| + \delta|S_3| \leq |G[S_1 \cup S_3, S_2]| \leq \Delta|S_2|$, we have $|S_1| + |S_3| \leq \frac{\Delta n + |S_1|}{\Delta+\delta}$.

Moreover, $S_1 \cup A$ is a total dominating set of G, so it follows that

$$\begin{aligned}\gamma_t(G) &\le |S_1 \cup A| \le |S_1| + \frac{1}{2}(|S_3| + \frac{|S_2|}{\delta}) \\ &= |S_1| + \frac{|S_3|}{2} + \frac{n - |S_1| - |S_3|}{2\delta} \\ &= \frac{n}{2\delta} + \frac{|S_1|}{2} + \frac{\delta - 1}{2\delta}(|S_1| + |S_3|) \\ &\le \frac{n}{2\delta} + \frac{|S_1|}{2} + \left(\frac{\delta - 1}{2\delta}\right)\left(\frac{\Delta n + |S_1|}{\Delta + \delta}\right) \\ &= \left(\frac{\delta(\Delta+\delta) + \delta - 1}{\delta(\Delta+\delta)}\right) im(G) + \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right)\end{aligned}$$

That is $\gamma_t(G) - \left(\frac{\delta(\Delta+\delta)+\delta-1}{\delta(\Delta+\delta)}\right) im(G) \le \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right)$.

Suppose the equality holds. Then all inequalities in the previous proof become equalities. It follows that S_2 is an independent set and $N(S_3) = S_2$. Furthermore, $d(v) = \Delta$ for each vertex $v \in S_2$ and $d(u) = \delta$ for each vertex $u \in S_1 \cup S_3$. We prove the following claims.

Claim 1. *For each vertex $v \in S_2$, $|N(v) \cap S_3| = 1$.*

Suppose to the contrary that there exists a vertex $v \in S_2$ such that $|N(v) \cap S_3| \ge 2$. Let $S_4 = S_3 - N(v) \cap S_3$ and $S_5 = S_2 - \{v\}$ and let H_1 be the bipartite subgraph of G with the partite sets (S_4, S_5) and the edge set defined by $G[S_4, S_5]$. By Lemma 2.1, there exists a set $B \subseteq S_5$ of cardinality at most $\frac{1}{2}\left(|S_4| + \frac{|S_5|}{\delta}\right)$ that dominates S_4. Since $S_1 \cup B \cup \{v\}$ is a total dominating set of G, it follows that

$$\begin{aligned}\gamma_t(G) &\le |S_1 \cup B \cup \{v\}| \le |S_1| + \frac{1}{2}(|S_4| + \frac{|S_5|}{\delta}) + 1 \\ &\le |S_1| + \frac{1}{2}(|S_3| - |N(v) \cap S_3|) + \frac{|S_2| - 1}{2\delta} + 1 \\ &\le |S_1| + \frac{1}{2}|S_3| + \frac{|S_2| - 1}{2\delta} < |S_1| + \frac{1}{2}|S_3| + \frac{|S_2|}{2\delta} \\ &= \left(\frac{\delta(\Delta+\delta) + \delta - 1}{\delta(\Delta+\delta)}\right) im(G) + \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right),\end{aligned}$$

which is a contradiction. Hence $|N(v) \cap S_3| \le 1$. Since $N(S_3) = S_2$, it follows that $|N(v) \cap S_3| = 1$.

Since $d(u) = \delta$ for each $u \in S_3$, by Claim 1 we have $|S_2| = \delta|S_3|$. Thus $\gamma_t(G) = |S_1| + |S_3|$.

Claim 2. *For each vertex $v \in S_1$, $|PN(v, S_1)| \ge 1$, where*

$$PN(v, S_1) = (N(v) \cap S_2) - [N(S_1 - \{v\}) \cap S_2].$$

Otherwise, if there exists a vertex $v \in S_1$ such that $|PN(v, S_1)| = 0$. Let $vu \in M$ for some $u \in V$, $w \in N(u) \cap S_2$, and a minimum subset $A \subseteq N(S_3)$ containing w

that dominates S_3. By Claim 1, we have $|A| = |S_3|$. Then $(S_1 - \{v\}) \cup A$ is a total dominating set of G, which contradicts $\gamma_t(G) = |S_1| + |S_3|$. So $|PN(v, S_1)| \geq 1$ for $v \in S_1$.

Moreover, it follows Claims 1 and 2 that $\delta = \Delta = 2$ and hence $G \cong C_5$.

Conversely, it is obvious that if G is isomorphic to either P_2 or C_5, then the equality holds. □

An edge incident with a leaf is called a *leaf edge*. A *pendant triangle* in a graph G is a triangle where two vertices of it are of degree 2 and the third vertex is of degree greater than 2.

Lemma 2.3. ([**1**]) *Let G be a connected graph. Then $\alpha'(G) = im(G)$ if and only if G is a star, C_3, or the graph obtained from a connected bipartite graph with bipartite vertex sets X and Y by attaching at least one leaf edge to each vertex of X, and possibly some pendant triangles to some vertices of Y.*

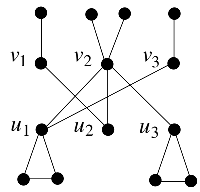

Figure 1. A connected graph G with $\alpha'(G) = im(G)$, where $X = \{v_1, v_2, v_3\}$ and $Y = \{u_1, u_2; u_3\}$.

By Theorem 2.2 and Lemma 2.3, we have the following corollary.

Corollary 2.4. *For any connected graph G,*

$$\gamma_t(G) - \alpha'(G) \leq \left(\frac{\Delta+1}{\Delta+\delta}\right)\left(\frac{n}{2}\right) + \left(\frac{\delta-1}{\delta(\Delta+\delta)}\right) im(G).$$

Furthermore equality holds if and only if G is isomorphic to either P_2 or C_3.

Remark. Suppose G is a connected graph with minimum degree δ. It is conjectured that the bound in Theorem 2.2 can be improved for large enough δ.

3. GRAPHS WITH TOTAL DOMINATION NUMBERS AT MOST OF THEIR MATCHING NUMBERS

3.1 Characterization of trees with total domination numbers at most of their matching numbers

A total dominating set of a graph G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set while a tree T with total domination number $\gamma_t(T)$ and matching number $\alpha'(T)$ is called a $(\gamma_t(T), \alpha'(T))$*-tree*. Before presenting our results, we make a couple of straightforward observations.

Observation 1. *If v is a support of a graph G, then v is in every $\gamma_t(G)$-set.*

Observation 2. *For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of G.*

Let T' be a $(\gamma_t(T'), \alpha'(T'))$-tree with $|V(T')| \geq 2$. For any $v \in V(T')$, let $T'_v = T' - \{v\}$. Define graphs $T(i,1)$ as in Figure 2 for $i = 2, 3, 4$, and $S(j,k)$ for $j = 3, 4, 8$ and $k \geq 1$ as in Figures 3. Let $S(5,k)$ denote the disjoint union of k copies of $T(3,1)$ and $S(6,k) = S(5,k)$. Let $S(7,k)$ denote the disjoint union of k copies of P_3. The vertex v_1 shown in the Figure 3 is called the *central vertex* of each of the graphs.

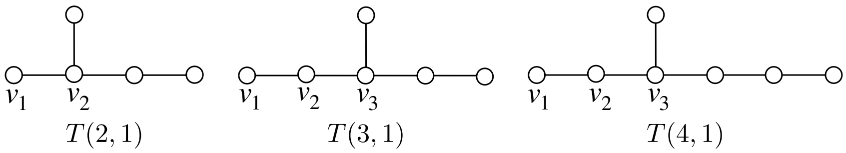

Figure 2.

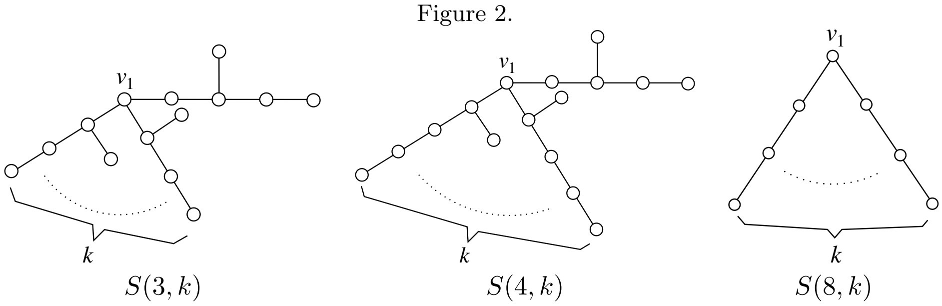

Figure 3.

Cockayne, Henning and Mynhardt [**3**] characterized the set of vertices of a tree that are contained in all (or in none) of, minimal total dominating sets of the tree as $D(T') = \{u \in V(T') \mid \text{there exists a } \gamma_t(T')\text{-set containing } u\}$ and $D'(T') = V(T') - D(T')$.

Lemma 3.1. *Let a tree T be obtained from a tree T' by joining a vertex v of T' to a leaf of P_4 (with an edge). Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Proof. Suppose that T is obtained from T' by joining v to a leaf x of path $P = xyzw$. Let S be a $\gamma_t(T')$-set of T'. Then it is obvious that $S \cup \{y, z\}$ is a total dominating set of T. So $\gamma_t(T) \leq \gamma_t(T') + 2$.

Let D be a $\gamma_t(T)$-set of T containing no leaves. Then $y, z \in D$. If $x \notin D$, then $D - \{y, z\}$ is a total dominating set of T' and $\gamma_t(T') \leq \gamma_t(T) - 2$. Suppose that $x \in D$. If $v \in D$, then $u \notin D$ for any vertex $u \in N_{T'}(v)$; otherwise, $D - \{x\}$ is a total dominating set of T, which is a contradiction. So $(D - \{x, y, z\}) \cup \{u\}$ is a total dominating set of T'. Hence, $\gamma_t(T') \leq \gamma_t(T) - 2$. If $v \notin D$, and there exists a vertex $u \in N_{T'}(v)$ such that $u \notin D$. Then $(D - \{x, y, z\}) \cup \{u\}$ is a total dominating set of T'. Hence, $\gamma_t(T') \leq \gamma_t(T) - 2$. Thus, we have $\gamma_t(T) = \gamma_t(T') + 2$.

Let M' be a maximum matching of T'. It is obvious that $M' \cup \{xy, zw\}$ is a matching of T. So, $\alpha'(T) \geq \alpha'(T') + 2$. Let M be a maximum matching of T saturating the largest number of vertices of P. Then $zw, xy \in M$. So $M - \{xy, zw\}$ is a matching of T'. Hence $\alpha'(T') \geq \alpha'(T) - 2$. Therefore $\alpha'(T) = \alpha'(T') + 2$. Hence we have $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$. □

From Lemma 3.2 to Lemma 3.8 below, we suppose that v is a vertex of a tree T'. By the similar way as in the previous proof, we have the following results.

Lemma 3.2. *Suppose that T'_v contains a component P_1. Let T be a tree obtained from T' by joining v to the vertex of P_1 or the leaf v_1 of $T(2,1)$ described in Figure 2. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.3. *Suppose that T'_v contains a component P_2. Let T be a tree obtained from T' by joining v to a leaf of P_2 or a support of P_4. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.4. *Suppose that T'_v contains a component P_3 such that v is adjacent to a leaf of P_3. Let T be obtained from T' by joining v to a support of P_4 or the leaf v_1 of $T(2,1)$. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.5. *Suppose that T'_v contains a component P_4 such that v is adjacent to a support of P_4. Let T be obtained from T' by joining v to a support of another P_4. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.6. *Suppose that T'_v contains a component $T(2,1)$ such that v is adjacent to the leaf v_1 of $T(2,1)$. Let T be obtained from T' by joining v to the leaf v_1 of another $T(2,1)$. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.7. *Suppose that T'_v contains a component P_5 such that v is adjacent to a support of P_5. Let T be obtained from T' by joining v to a support of P_4. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Lemma 3.8. *Suppose that T'_v contain components P_1 and P_3 such that v is adjacent to a leaf of P_3. Let T be obtained from T' by joining v to a vertex of P_2. Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.*

Let v be a vertex of T, and ζ be a family of trees such that each tree $T \in \zeta$ has the following properties:

(1) Let $C(T) = \{u \in V(T) \mid d(u) \geq 3\}$. For any $u \in C(T)$, T_u does not contain a component P_t $(t \geq 4)$ with a leaf adjacent to u.

(2) If one of the components of T_v is P_1, then other components of T_v are neither P_1, nor $T(2,1)$ with the leaf v_1 adjacent to v.

(3) If one of the components of T_v is P_2, then other components of T_v are neither P_2, nor P_4 with a support adjacent to v.

(4) If P_3 with a leaf adjacent to v is a component of T_v, then no other components of T_v are P_4 with a support adjacent to v, or $T(2,1)$ with the leaf v_1 adjacent to v.

(5) At most one component of T_v is P_4 with a support adjacent to v.

(6) At most one component of T_v is $T(2,1)$ with the leaf v_1 of $T(2,1)$ is adjacent to v.

(7) If one of the components of T_v is P_5 with a support adjacent to v, then no other components of T_v are P_4 with a support adjacent to v.

(8) If P_1, and P_3 with a leaf adjacent to v are components of T_v, then P_2 is not a component of T_v.

For any tree T, by Lemmas 3.1 to 3.8, either $T \in \zeta$ or T can be transformed into some tree $T' \in \zeta$ such that $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$. In this situation, we say that T is an *extension* of T'. Thus, in order to give a characterization of the trees with $\gamma_t(T) \leq \alpha'(T)$, we define the following operations on trees.

Suppose that a tree T is obtained from another tree T' by the following operations.

Operation 1. T is obtained from tree $T' \in \zeta$ as an extension.

Operation 2. Suppose that T'_v contains a component $P_1 = \{w\}$ and $P_2 = xy$ such that $vx \in E(T')$, where $v \in V(T')$. Join x to a leaf of another P_2 or join w to a support of P_4.

Operation 3. Join the central vertex v_1 of $S(3,k)$ to a vertex of T' for some k.

Operation 4. Join the central vertex v_1 of $S(4,k)$ to a vertex of T' for some k.

Operation 5. For each $u \in D(T')$, attach $S(5,k)$ by joining each vertex v_1 to u for some k.

Operation 6. For each $u \in D'(T')$, attach $S(6,k)$ by joining each vertex v_1 to u for some k.

Operation 7. Suppose that T'_v contains components $P_1 = \{w\}$ and $P_2 = xy$ such that $vx \in E(T')$. Delete the component P_1 or P_2 and attach $S(7,k)$ by joining a leaf of each P_3 to v, for some integer k.

Operation 8. For each $v \in V(T')$, attach $S(8,k)$ by joining vertex v_1 to v for some k.

Since the following lemmas can be obtained in a similar way as Lemma 3.1, their proofs are omitted.

Lemma 3.9. *Suppose that T is obtained from T' by operation* 2. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T')$.

Lemma 3.10. *Suppose that T is obtained from T' by operation* 3. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') - 1$.

Lemma 3.11. *Suppose that T is obtained from T' by operation* 4. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') + k - 1$.

Lemma 3.12. *Suppose that* T *is obtained from* T' *by operation* 5. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') - k$.

Lemma 3.13. *Suppose that* T *is obtained from* T' *by operation* 6. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') - k + 1$.

Lemma 3.14. *Suppose that* T *is obtained from* T' *by operation* 7. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') + k$.

Lemma 3.15. *Suppose that* T *is obtained from* T' *by operation* 8. *Then* $\gamma_t(T) - \alpha'(T) = \gamma_t(T') - \alpha'(T') + k - 1$.

Let $c(i)$ denote the number of operations i required to construct the tree T from P_2 or P_4 for $i = 1, 2, \ldots, 8$. For each operation i, assume that $S(i, k_{ij})$ is attached for some integer k_{ij}, where $j = 1, 2, \ldots, c(i)$ and $i = 3, 4, 5, 6, 7, 8$.

Theorem 3.16. *Suppose that* T *is a tree of order* n *for* $n \geq 3$. *Then* T *can be obtained from a path* P_ℓ *by a finite sequence of operations* i *for* $i = 1, 2, \ldots, 8$, *where* $l = 2$ *or* 4. *Furthermore* $\gamma_t(T) - \alpha'(T) = \gamma_t(P_\ell) - \alpha'(P_\ell) - c(3) + \sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)} k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)} k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1)$.

Proof. We proceed by induction on the order n of T. If $diam(T) = 2$, then T is a star. So T is obtained from P_2 by operation 1. If $diam(T) = 3$, then T is a double star. So T is obtained from P_4 by operation 1. If T is isomorphic to $P_5, T(3, 1)$ or $T(4, 1)$, then it is obvious that the result holds. Assume that every tree T' of order $5 \leq n' < n$ can be obtained from P_2 or P_4 by a finite sequence of operations i for $i = 1, 2, \ldots, 8$.

Let T be a tree of order n such that T is not isomorphic to $T(3, 1)$ and $T(4, 1)$. Assume that the longest path P of T is $u_1 u_2 \cdots u_t$. Without loss of generality, we can assume that $t \geq 5$. If $T \notin \zeta$, then T can be obtained from some tree $T' \in \zeta$ by operation 1. Without loss of generality, we can assume that $T \in \zeta$. Then $d(u_1) = 1, d(u_2) = 2$ and $2 \leq d(u_3) \leq 3$, and we may have the following cases.

Case 1. $d(u_3) = 3$. Then u_3 is a support of T. Let $N(u_3) - \{u_2, u_4\} = \{u_3'\}$. If $d(u_4) \geq 3$, then $d(u_4) = 3$ and u_4 is a support of T. Let $T' = T - \{u_1, u_2\}$. So T is obtained from T' by operation 2. Without loss of generality, we can assume that $d(u_4) = 2$. Since T is not isomorphic to $T(3, 1)$, it follows that $d(u_5) \geq 2$.

Case 1.1. $d(u_5) \geq 3$.

Let $T_{u_5 1}, T_{u_5 2}, \ldots, T_{u_5 d(u_5)}$ denote components of $T_{u_5} = T - \{u_5\}$ such that $u_1 \in T_{u_5 1}$ and $u_t \in T_{u_5 d(u_5)}$. Since $T \in \zeta$, it follows that $T_{u_5 i}$ is isomorphic to P_2, P_4 or P_5 for $i = 2, \ldots, d(u_5) - 1$, where one support of each P_4 and P_5 is adjacent to u_5.

If there exists i such that $T_{u_5 i}$ is isomorphic to P_2 for $i = 2, \ldots, d(u_5) - 1$, then let $T' = T - \{u_1, u_2, u_3, u_3'\}$. So T is obtained from T' by operation 2.

Now assume T_{u_5i} is not isomorphic to P_2 for any $i = 2, \dots, d(u_5) - 1$.

If there exists i such that T_{u_5i} is isomorphic to P_5 for $i = 2, \dots, d(u_5) - 1$, then T_{u_5i} is P_5 for $i = 2, \dots, d(u_5) - 1$. Let $T' = T_{u_5d(u_5)}$. It follows that T is obtained from T' by operation 4.

If T_{u_5i} is neither isomorphic to P_2 nor P_5 for any $i = 2, \dots, d(u_5) - 1$, then T_{u_5i} is P_4 for $i = 2, \dots, d(u_5) - 1$. Let $T' = T_{u_5d(u_5)}$. It follows that T is obtained from T' by operation 3.

Case 1.2. $d(u_5) = 2$. Since T is not isomorphic to $T(4, 1)$, it follows that $d(u_6) \geq 2$. Assume that T_{u_6} has k components $T(3, 1)$. Let $T_{u_61}, T_{u_62}, \dots, T_{u_6k}$ denote components $T(3, 1)$ of T_{u_6}. If $k = d(u_6)$, let $T' = T - T_{u_61} - T_{u_62} - \cdots - T_{u_6(k-1)}$; otherwise, $T' = T - T_{u_61} - T_{u_62} - \cdots - T_{u_6k}$. Thus, T is obtained from T' by operation 5 or 6.

Case 2. $d(u_3) = 2$. Suppose that $d(u_4) = 2$. Since T is not isomorphic to P_5, it follows that $d(u_5) \geq 2$. Let $T' = T - \{u_1, u_2, u_3, u_4\}$. So T is obtained from T' by operation 1. Without loss of generality, assume that $d(u_4) \geq 3$.

Let $T_{u_41}, T_{u_42}, \dots, T_{u_4d(u_4)}$ denote the components of $T_{u_4} = T - \{u_4\}$ such that $u_1 \in T_{u_41}$ and $u_t \in T_{u_4d(u_4)}$. Then T_{u_4i} is isomorphic to P_1, P_2 or P_3 for $i = 2, \dots, d(u_4) - 1$, where one leaf of each P_3 is adjacent to u_4.

If there exists i such that T_{u_4i} is isomorphic to P_1 for $i = 2, \dots, d(u_4) - 1$, say $T_{u_4(d(u_4)-1)} = P_1$, then T_{u_4i} is P_3 for $i = 1, 2, \dots, d(u_4) - 2$. Let

$$T' = T - T_{u_42} - \cdots - T_{u_4(d(u_4)-2)} - \{u_1\}.$$

So T is obtained from T' by operation 7.

Now assume T_{u_4i} is not isomorphic to P_1 for any $i = 2, \dots, d(u_4) - 1$.

If there exists i such that T_{u_4i} is isomorphic to P_2 for $i = 2, \dots, d(u_4) - 1$, say $T_{u_4(d(u_4)-1)} = P_2$, then T_{u_4i} is P_3 for $i = 1, 2, \dots, d(u_4) - 2$. Let

$$T' = T - T_{u_42} - \cdots - T_{u_4(d(u_4)-2)} - \{u_1, u_2\}.$$

So T is obtained from T' by operation 7.

If T_{u_4i} is neither isomorphic to P_1 nor P_2 for any $i = 2, \dots, d(u_4) - 1$, then T_{u_4i} is P_3 for $i = 1, 2, \dots, d(u_4) - 1$. Let $T' = T_{u_4d(u_4)}$. It follows that T is obtained from T' by operation 8.

By Cases 1 and 2, any tree T can be obtained from T' by operation i for $i = 1, 2, \dots, 8$. Since $|V(T')| < n$, T' can be obtained from a path P_2 or P_4 by a finite sequence of operations i for $i = 1, 2, \dots, 8$. It follows that T can be obtained from P_2 or P_4 by a finite sequence of operations i for $i = 1, 2, \dots, 8$.

Let $c'(i)$ denote the number of operations i required to construct the tree T' from P_ℓ, $\ell = 2, 4$, for $i = 1, 2, \dots, 8$. For each operation i, $S(i, k'_{ij})$ is attached, where $j = 1, 2, \dots, c'(i)$ and $i = 3, 4, 5, 6, 7, 8$. So, $\gamma_t(T') - \alpha'(T') = \gamma_t(P_\ell) - \alpha'(P_\ell)$ $-c'(3) + \sum_{j=1}^{c'(4)} (k'_{4j} - 1) - \sum_{j=1}^{c'(5)} k'_{5j} - \sum_{j=1}^{c(6)} (k'_{6j} - 1) + \sum_{j=1}^{c'(7)} k'_{7j} + \sum_{j=1}^{c'(8)} (k'_{8j} - 1)$.

Let $c(i)$ denote the number of operations i required to construct the tree T from P_ℓ, $\ell = 2, 4$, for $i = 1, 2, \dots, 8$. For each operation i, $S(i, k_{ij})$ is attached, where $j = 1, 2, \ldots, c(i)$ and $i = 3, 4, 5, 6, 7, 8$. Since T is obtained from T' by some operation h, it follows that $c(h) = c'(h) + 1$, and $c(j) = c'(j)$ for $j \neq h$. Furthermore, $k_{hj} = k'_{hj}$ for $j = 1, \ldots, c'(h)$, and $k_{tj} = k'_{tj}$ for $j = 1, \ldots, c(t)$ and $t \neq h$. By Lemmas 3.9–3.15, it follows that $\gamma_t(T) - \alpha'(T) = \gamma_t(P_\ell) - \alpha'(P_\ell) - c(3)$ $+\sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)} k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)} k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1)$. □

Corollary 3.17. *If a tree T can be obtained from path P_2 by a finite sequence of operations i for $i = 1, 2, \ldots, 8$, then $\gamma_t(T) \leq \alpha'(T)$ if and only if* $\sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)} k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)} k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1) \leq c(3) - 1$.

Corollary 3.18. *If a tree T can be obtained from path P_4 by a finite sequence of operations i for $i = 1, 2, \ldots, 8$, then $\gamma_t(T) \leq \alpha'(T)$ if and only if* $\sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)} k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)} k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1) \leq c(3)$.

3.2 A family of graphs with total domination numbers at most of their matching numbers

A family of graphs with total domination number at most their matching number will be given in the following. Let $\eta = \{T \mid \gamma_t(T) \leq \alpha'(T)\}$. Define a family of graphs ϱ. A graph $G \in \varrho$ if and only if G contains a spanning tree $T \in \eta$.

Lemma 3.19. *Let G be a connected graph. If G contains a spanning tree $T \in \eta$. Then $\gamma_t(G) \leq \alpha'(G)$.*

Proof. We will prove by induction on the number of edges of G. If $|E(G)| = n - 1$, then G is a tree and $G \in \eta$. So $\gamma_t(G) \leq \alpha'(G)$. Suppose that the property is true for all graph with the number of edges less than k. Let G be a connected graph with k edges and $k > n - 1$. Suppose that $T \in \eta$ is a spanning tree of G. Let $e \in E(G) - E(T)$ and $G' = G - e$. Then T is also a spanning tree of G'. By the induction hypothesis, we have $\gamma_t(G') \leq \alpha'(G')$. It is obvious that $\gamma_t(G) \leq \gamma_t(G')$ and $\alpha'(G') \leq \alpha'(G)$. Hence $\gamma_t(G) \leq \alpha'(G)$. □

By Lemma 3.19 we have the following result.

Theorem 3.20. *For any graph $G \in \varrho$, $\gamma_t(G) \leq \alpha'(G)$.*

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