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STAIRCASE WORDS AND CHEBYSHEV POLYNOMIALS

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A word $\sigma = \sigma_1 \cdots \sigma_n$ over the alphabet $[k] = \{1, 2, \ldots, k\}$ is said to be a *staircase* if there are no two adjacent letters with difference greater than 1. A word σ is said to be *staircase-cyclic* if it is a staircase word and in addition satisfies $|\sigma_n - \sigma_1| \leq 1$. We find the explicit generating functions for the number of staircase words and staircase-cyclic words in $[k]^n$, in terms of *Chebyshev polynomials of the second kind*. Additionally, we find explicit formulæ for the numbers themselves, as trigonometric sums. These lead to immediate asymptotic corollaries. We also enumerate staircase necklaces, which are staircase-cyclic words that are not equivalent up to rotation.

1. INTRODUCTION

Let $[k]^n$ be the set of all words of length n over the alphabet $[k] = \{1, 2, \ldots, k\}$. We consider the enumeration of a class of words with constrained variation namely the *staircase words*. A word $w \in [k]^n$ is said to be a staircase (or staircase word) if the absolute difference between each pair of adjacent letters of w is at most 1. Intuitively such words have the "nice" property that one can move between adjacent letters of a word by taking at most one staircase step at a time: one step upward, one step downward, or remain at the same level. Staircase words can also be interpreted as a certain class of Motzkin paths (with steps $(1, 1)$, $(1, -1)$ and $(1, 0)$) that lie in a strip with heights bounded by 0 to $k - 1$.

The enumeration of words which contain a prescribed number of a given set of strings as substrings is a classical problem in combinatorics. This problem can, for example, be attacked using the transfer matrix method, see [**11**, Section 4.7] and [**3**]. In recent times, many research papers have been devoted to the study of

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enumeration problems on the set $[k]^n$ under various pattern constraints. In this connection we mention the works of Guibas and Odlyzko (see [**4**, **5**]) which are devoted to the analysis of repetitive patterns in random words as well as various pattern matching techniques in general words.

The pattern approach to enumeration of words gained prominence following the work of Regev [**7**] who provided explicit and asymptotic formulæ for the number words counted by strictly descending substrings of given length. Subsequently, Burstein's thesis [**1**] undertook a systematic development and found enumeration results on words avoiding several classes of patterns. Régnier and Szpankowski [**9**] used a combinatorial approach to study the frequency of occurrences of certain strings (which they also call a "pattern") in a random word, where overlapping copies of the strings are counted separately. Burstein and Mansour [**2**] have also considered the enumeration of elements of $[k]^n$ that satisfy certain restrictions characterized in terms of pattern avoidance.

The staircase property may, of course, be characterized in the language of pattern avoidance. A word $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in [k]^n$ possesses the staircase property provided it avoids a (contiguous) string of the form ij where $|i-j| > 1$. Thus a staircase word is any word that fulfills the staircase property. For example, there are 7 staircase words in $[3]^2$, namely 11, 12, 21, 22, 23, 32 and 33. A word σ is said to be *staircase-cyclic* if it is a staircase and in addition satisfies $\sigma_n - \sigma_1 \in \{0, 1, -1\}$. Clearly, each word in $[k]^n$, $k = 1, 2$, is staircase-cyclic (and thus also a staircase). We denote the number of staircase words (respectively, staircase-cyclic words) in $[k]^n$ by $sw_{n,k}$ (respectively, $scw_{n,k}$). Table 1 shows the numbers of staircase words and staircase-cyclic words of length n over the alphabet $[k]$ for $0 \le n \le 11$ and $2 \le k \le 7$. On comparing the sequences with those in the On-Line Encyclopedia of Integer Sequences [**8**] we find the equivalent definition of $sw_{n,k}$ as the number of base k n-digit numbers with adjacent digits differing by one or less. However, when $k > 3$ the entries in [**8**] contain contain no information on computational formulae for the sequences.

n	0	1	2	3	4	5	6	7	8	9	10	11
$sw_{n,2}$	1	2	4	8	16	32	64	128	256	512	1024	2048
$scw_{n,2}$	1	2	4	8	16	32	64	128	256	512	1024	2048
$sw_{n,3}$	1	3	7	17	41	99	239	577	1393	3363	8119	19601
$scw_{n,3}$	1	3	7	15	35	83	199	479	1155	2787	6727	16239
$sw_{n,4}$	1	4	10	26	68	178	466	1220	3194	8362	21892	57314
$scw_{n,4}$	1	4	10	22	54	134	340	872	2254	5854	15250	39802
$sw_{n,5}$	1	5	13	35	95	259	707	1931	5275	14411	39371	107563
$scw_{n,5}$	1	5	13	29	73	185	481	1265	3361	8993	24193	65345
$sw_{n,6}$	1	6	16	44	122	340	950	2658	7442	20844	58392	163594
$scw_{n,6}$	1	6	16	36	92	236	622	1658	4468	12132	33146	90998
$sw_{n,7}$	1	7	19	53	149	421	1193	3387	9627	27383	77923	221805
$scw_{n,7}$	1	7	19	43	111	287	763	2051	5575	15271	42099	116651

Table 1. Numbers of staircase words and staircase-cyclic words $sw_{n,k}, scw_{n,k}$.

We will find the explicit generating functions for $sw_{n,k}$ and $scw_{n,k}$ in terms of *Chebyshev polynomials of the second kind.* Additionally, we find explicit formulæ for the numbers themselves, in terms of certain trigonometric expressions. They allow for immediate asymptotic corollaries.

Chebyshev polynomials of the second kind are defined by

$$U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$$

for $r \geq 0$. Evidently, $U_r(x)$ is a polynomial of degree r in x with integer coefficients. For example, $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and in general,

$$U_r(x) = 2xU_{r-1}(x) - U_{r-2}(x). \tag{1.1}$$

Chebyshev polynomials of the first kind are defined by $T_r(\cos\theta) = \cos(r\theta)$ which is equivalent to $T_r(x) = \frac{1}{2}\big(U_r(x) - U_{r-2}(x)\big)$. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [**10**]).

Two words $\sigma = \sigma_1 \cdots \sigma_n$ and $\pi = \pi_1 \cdots \pi_n$ in $[k]^n$ are said to be *rotation equivalent* if there exists an index i, $1 \leq i \leq n$ such that $\pi_i\pi_{i+1} \cdots \pi_n\pi_1\pi_2 \cdots \pi_{i-1} = \sigma$. For example, the words 122, 212 and 221 are rotation equivalent. The set of *necklaces* of length n over the alphabet $[k]$ is the set of words in $[k]^n$ up to the rotation-equivalence. For example, if $k = 2$ and $n = 3$ there are 4 necklaces, namely, 111, 122 (212 and 221 are rotation equivalent), 112 (121 and 211 are rotation equivalent) and 222. Using our results on staircase-cyclic words, we also determine the number of staircase necklaces in $[k]^n$.

The paper is organized as follows. In Section 2 we obtain the generating function for the number $sw_{n,k}$ of staircase words, followed shortly by the explicit enumeration formula (Theorems 2.2 and 2.4). The asymptotic growth rate is also obtained in this section. In Section 3 we obtain the corresponding enumeration results for staircase-cyclic words. Lastly, Section 4 deals with the enumeration of staircase necklaces.

2. ENUMERATION OF STAIRCASE WORDS

Let $sw_k(x)$ denote the generating function for the number of staircase words over $[k]$:

$$sw_k(x) = \sum_{n\geq 0} sw_{n,k}x^n.$$

In order to obtain a formula for $sw_k(x)$, we introduce the following notations. Let $sw_k(x \mid i_1i_2\cdots i_s)$ be the generating function for the number of staircase words $\sigma_1\cdots\sigma_n$ of length n over the alphabet $[k]$ such that $\sigma_1\cdots\sigma_s = i_1\cdots i_s$.

Lemma 2. *The generating function $sw_k(x \mid i)$ satisfies*

$$sw_k(x \mid i) = x + x\big(sw_k(x \mid i-1) + sw_k(x \mid i) + sw_k(x \mid i+1)\big),$$

for all $1 \le i \le k$*, where* $sw_k(x \mid i) = 0$ *if* $i \notin [k]$.

Proof. Let σ be any nonempty staircase word. If σ contains exactly one letter then $\sigma_1 \in [k]$. Otherwise, the second letter of $\sigma = i\sigma_2 \cdots \sigma_n$ is either $i-1, i$ or $i+1$. Thus, in terms of generating functions we have that

$$sw_k(x \mid i) = x + x\big(sw_k(x \mid i-1) + sw_k(x \mid i) + sw_k(x \mid i+1)\big),$$

for all $1 \le i \le k$, which completes the proof. □

Rewriting Lemma 2.1 as a matrix system we obtain

$$\mathbf{A}\begin{pmatrix} sw_k(x \mid 1) \\ \vdots \\ sw_k(x \mid k) \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} x, \tag{2.1}$$

where $\mathbf{A} = (a_{ij})$ is a $k \times k$ matrix defined by $a_{ii} = 1 - x$, $a_{i(i+1)} = a_{(i+1)i} = -x$, and $a_{ij} = 0$ for all $|i - j| > 1$. Clearly, $\mathbf{A}$ is a tridiagonal matrix.

Applying a result of Usmani [**13**] or equivalently [**6**], on the inversion of $\mathbf{A}$ we get

$$(\mathbf{A}^{-1})_{ij} = \begin{cases} x^{j-i}\theta_{i-1}\theta_{k-j}/\theta_k & i \le j, \\ x^{i-j}\theta_{j-1}\theta_{k-i}/\theta_k & i > j, \end{cases}$$

where θ_i satisfies the recurrence relation $\theta_i = (1-x)\theta_{i-1} - x^2\theta_{i-2}$, with the initial conditions $\theta_0 = 1$ and $\theta_1 = 1 - x$. It follows from (1.1) that the solution is given by $\theta_i = x^i U_i\left(\frac{1-x}{2x}\right)$. Consequently

$$(\mathbf{A}^{-1})_{ij} = \begin{cases} \dfrac{U_{i-1}\left(\frac{1-x}{2x}\right)U_{k-j}\left(\frac{1-x}{2x}\right)}{xU_k\left(\frac{1-x}{2x}\right)} & i \le j, \\ \dfrac{U_{j-1}\left(\frac{1-x}{2x}\right)U_{k-i}\left(\frac{1-x}{2x}\right)}{xU_k\left(\frac{1-x}{2x}\right)} & i > j. \end{cases} \tag{2.2}$$

Thus the solution of (2.1) is

$$\begin{pmatrix} sw_k(x \mid 1) \\ \vdots \\ sw_k(x \mid k) \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} x.$$

This implies that the generating function $sw_k(x \mid i)$ is given by

$$sw_k(x \mid i) = \frac{1}{U_k(t)}\left[U_{k-i}(t)\sum_{j=0}^{i-2} U_j(t) + U_{i-1}(t)\sum_{j=0}^{k-i} U_j(t)\right],$$

where $t = \dfrac{1-x}{2x}$.

In order to simplify the right-hand side we use the identity

$$\sum_{j=0}^{p} U_j(t) = \frac{U_{p+1}(t) - U_p(t) - 1}{2(t-1)}, \tag{2.3}$$

which may be proved easily from the fact that

$$\sum_{j=0}^{p} \sin(jt) = \frac{\sin((p+1)t)(\cos(t)-1) + \sin(t)\cos((p+1)t) - \sin(t)}{2(\cos(t)-1)}$$

and $T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x))$. The verification of such identities is *a priori* trivial and can be done by a computer, since, upon rewriting the trigonometric functions via Euler's formulæ, one only has to sum some finite geometric series. Thus

$$sw_k(x \mid i) = \frac{1}{2(t-1)U_k(t)}\left[U_{i-1}(t)U_{k+1-i}(t) - U_{k-i}(t)U_{i-2}(t) - U_{k-i}(t) - U_{i-1}(t)\right].$$

Now apply the identity

$$U_i(t)U_j(t) = \frac{U_{i-j}(t) - tU_{i-j-1}(t) - U_{i+j+2}(t) + tU_{i+j+1}(t)}{2(1-t^2)}, \tag{2.4}$$

to obtain

$$sw_k(x \mid i) = \frac{1}{2(t-1)U_k(t)}\left[\frac{U_k(t) - tU_{k-1}(t) - U_{k+2}(t) + tU_{k+1}(t)}{2(1-t^2)} - U_{k-i}(t) - U_{i-1}(t)\right],$$

which, by (1.1), is equivalent to

$$sw_k(x \mid i) = \frac{1}{2(t-1)U_k(t)}\left[U_k(t) - U_{k-i}(t) - U_{i-1}(t)\right]. \tag{2.5}$$

Now, using $sw_k(x) = 1 + \sum_{i=1}^{k} sw_k(x \mid i)$ and again (2.3), we obtain an explicit formula for the generating function $sw_k(x)$.

Theorem 2.2. *The generating function $sw_k(x)$ for the number of staircase words of length n over the alphabet $[k]$ is given by*

$$sw_k(x) = 1 + \frac{x(k-(3k+2)x)}{(1-3x)^2} + \frac{2x^2}{(1-3x)^2}\,\frac{1 + U_{k-1}\left(\frac{1-x}{2x}\right)}{U_k\left(\frac{1-x}{2x}\right)}.$$

It is easy to see that each word in $[k]^n$, $k = 1, 2$, is a staircase. For small values of k, Theorem 2.2 gives

- $sw_3(x) = \dfrac{1+x}{1-2x-x^2}$, that is, the number of smooth words in $[3]^n$ is given by
$$\frac{1}{2}\big(1+\sqrt{2}\,\big)^{n+1} + \frac{1}{2}\big(1-\sqrt{2}\,\big)^{n+1}.$$

- $sw_4(x) = \dfrac{1+x-x^2}{1-3x+x^2}$, that is, the number of staircase words in $[4]^n$ is given by
$$\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} = 2F_{2n+1}.$$

- $sw_5(x) = \dfrac{1+2x-2x^2-2x^3}{(1-x)(1-2x-2x^2)}$, that is, the number of staircase words in $[5]^n$ is given by
$$\frac{2+\sqrt{3}}{6}\big(1+\sqrt{3}\,\big)^{n+1} + \frac{2-\sqrt{3}}{6}\big(1-\sqrt{3}\,\big)^{n+1} + \frac{1}{3}\,.$$

In order to obtain an explicit formula for the number of staircase words of length n over the alphabet $[k]$ we need the following lemma.

Lemma 2.3. *Let $m \geq 1$. Then*
$$\frac{1}{U_m(x)} = \frac{1}{m+1}\sum_{j=1}^{m} \frac{(-1)^{j+1}\sin^2\left(\frac{j\pi}{m+1}\right)}{x-\cos\left(\frac{j\pi}{m+1}\right)}$$
and
$$\frac{1+U_{m-1}(x)}{U_m(x)} = \frac{1}{m+1}\sum_{j=1}^{m} \frac{(1+(-1)^{j+1})\sin^2\left(\frac{j\pi}{m+1}\right)}{x-\cos\left(\frac{j\pi}{m+1}\right)}\,.$$

Proof. Let us compute the partial fraction decomposition of $\dfrac{1}{U_m(x)}$. By general principles, it is $\displaystyle\sum_{j=1}^{m}\frac{A_{m,j}}{x-\rho_{m,j}}$, where $\rho_{m,j} = \cos\left(\dfrac{j\pi}{m+1}\right)$ are the zeros of the m-th Chebyshev polynomials of the second kind. Now, $A_{m,j} = \dfrac{1}{U_m'(\rho_{m,j})}$ and note that
$$U_m'(x) = \frac{\mathrm{d}U_m(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\frac{\sin(m+1)\theta}{\sin\theta}\Big)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x}\,.$$
We work out that
$$\frac{\mathrm{d}U_m}{\mathrm{d}\theta} = \frac{(m+1)\cos(m+1)\theta\cdot\sin\theta - \sin(m+1)\theta\cdot\cos\theta}{\sin^2\theta}\,,$$

and if we plug in $x = \rho_{m,j}$ simplification occurs, since certain terms are just zero; we obtain that

$$\begin{aligned}\frac{\mathrm{d}U_m}{\mathrm{d}\theta}(\arccos \rho_{m,j}) &= \frac{(m+1)\cos(m+1)\theta \cdot \sin\theta - \sin(m+1)\theta \cdot \cos\theta}{\sin^2\theta}\Bigg|_{\theta = \frac{\pi j}{m+1}} \\ &= \frac{(m+1)\cos(\pi j)\cdot \sin\dfrac{\pi j}{m+1} - \sin(\pi j)\cdot\cos\dfrac{\pi j}{m+1}}{\sin^2\dfrac{\pi j}{m+1}} \\ &= \frac{(m+1)\cos(\pi j)\cdot\sin\dfrac{\pi j}{m+1}}{\sin^2\dfrac{\pi j}{m+1}} = \frac{(m+1)(-1)^j}{\sin\dfrac{\pi j}{m+1}}.\end{aligned}$$

Further $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$, so together $\dfrac{1}{A_{m,j}} = U_m'(\rho_{m,j}) = \dfrac{(m+1)(-1)^{j+1}}{\sin^2\dfrac{\pi j}{m+1}}$, which completes the proof of the first identity. Similarly, the second one can be obtained.□

We are ready to obtain an explicit formula.

Theorem 2.4. *The number of staircase words of length n over the alphabet $[k]$ is given by*

$$sw_{n,k} = \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2\frac{j\pi}{2(k+1)}\Big(1+2\cos\frac{j\pi}{k+1}\Big)^{n-1},$$

or, alternatively, as

$$sw_{n,k} = \frac{2}{k+1}\sum_{0\le j\le \frac{k-1}{2}}\cot^2\frac{(2j+1)\pi}{2(k+1)}\Big(1+2\cos\frac{(2j+1)\pi}{k+1}\Big)^{n-1}.$$

Proof. Fix k and let $\theta_j = \dfrac{j\pi}{k+1}$. Lemma 2.3 says that the coefficient of x^n in $\dfrac{2x^2}{(1-3x)^2}\dfrac{1+U_{k-1}(t)}{U_k(t)}$ with $t = \dfrac{1-x}{2x}$ (see Theorem 2.2) is given by

$$p_{n,k} = [x^n]\,\frac{2x^2}{(1-3x)^2}\,\frac{1}{k+1}\sum_{j=1}^{k}\frac{(1+(-1)^{j+1})\sin^2\theta_j}{\dfrac{1-x}{2x}-\cos\theta_j} \tag{2.6}$$

$$= \frac{4}{k+1}\,[x^{n-3}]\,\frac{1}{(1-3x)^2}\sum_{j=1}^{k}\frac{(1+(-1)^{j+1})\sin^2\theta_j}{1-x\,(1+2\cos\theta_j)}.$$

Now notice that

$$\frac{1}{(1-3x)^2(1-x\omega)} = -\frac{3}{(\omega-3)(1-3x)^2} - \frac{3\omega}{(\omega-3)^2(1-3x)} + \frac{\omega^2}{(\omega-3)^2(1-x\omega)}$$

and

$$\omega - 3 = 1 + 2\cos\theta_j - 3 = -4\sin^2(\theta_j/2). \tag{2.7}$$

So we are dealing with

$$A = \frac{3}{4\sin^2(\theta_j/2)(1-3x)^2} - \frac{3\,(1+2\cos\theta_j)}{16\sin^4(\theta_j/2)(1-3x)} + \frac{(1+2\cos\theta_j)^2}{16\sin^4(\theta_j/2)(1-x(1+2\cos\theta_j))},$$

which implies that the coefficient of x^{n-3} in A is given by

$$\frac{3^{n-2}(n-2)}{4\sin^2(\theta_j/2)} - \frac{3^{n-2}(1+2\cos\theta_j)}{16\sin^4(\theta_j/2)} + \frac{(1+2\cos\theta_j)^{n-1}}{16\sin^4(\theta_j/2)}\,.$$

Hence, (2.6) can be written as

$$\begin{aligned} p_{n,k} = \frac{4}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\sin^2\theta_j \Bigg[& \frac{3^{n-2}(n-2)}{4\sin^2(\theta_j/2)} \\ & - \frac{3^{n-2}(1+2\cos\theta_j)}{16\sin^4(\theta_j/2)} + \frac{(1+2\cos\theta_j)^{n-1}}{16\sin^4(\theta_j/2)}\Bigg] \end{aligned}$$

which simplifies to

$$\begin{aligned} p_{n,k} = \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2(\theta_j/2)\Bigg[& 4(n-2)3^{n-2} \\ & - \frac{3^{n-2}(1+2\cos\theta_j)}{\sin^2(\theta_j/2)} + \frac{(1+2\cos\theta_j)^{n-1}}{\sin^2(\theta_j/2)}\Bigg]. \end{aligned}$$

Using the identity $\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2(\theta_j/2) = \frac{k+1}{2}$, we get that

$$\begin{aligned} p_{n,k} = 2(n-2)3^{n-2} + \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2(\theta_j/2)\Big[& (1+2\cos\theta_j)^{n-1} \\ & - 3^{n-2}(1+2\cos\theta_j)\Big]. \end{aligned}$$

Note that

$$\begin{aligned} [x^n]\frac{x(k-(3k+2)x)}{(1-3x)^2} &= [x^{n-1}]\frac{k}{(1-3x)^2} - [x^{n-2}]\frac{3k+2}{(1-3x)^2} \\ &= kn3^{n-1} - (n-1)(3k+2)3^{n-2} \\ &= (3k-2n+2)3^{n-2}. \end{aligned}$$

Therefore, Theorem 2.2 gives

$$\begin{aligned} sw_{n,k} &= (3k-2n+2)3^{n-2} + p_{n,k} \\ &= (3k-2)3^{n-2} + \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2(\theta_j/2)\Big[(1+2\cos\theta_j)^{n-1} \\ &\qquad\qquad\qquad\qquad\qquad - 3^{n-2}(1+2\cos\theta_j)\Big] \\ &= (3k-2)3^{n-2} + \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2(\theta_j/2)(1+2\cos\theta_j)\times \\ &\qquad\qquad\qquad\qquad\qquad \times\Big[(1+2\cos\theta_j)^{n-2} - 3^{n-2}\Big]. \end{aligned}$$

Notice further that

$$\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2(\theta_j/2)(1+2\cos\theta_j) = (k+1)(3k-2),$$

thus we have that

$$sw_{n,k} = \frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2(\theta_j/2)(1+2\cos\theta_j)^{n-1},$$

as claimed. □

Corollary 2.5. *Asymptotically, we have as* $n\to\infty$,

$$sw_{n,k} \sim \frac{2}{k+1}\cot^2\frac{\pi}{2(k+1)}\Big(1+2\cos\frac{\pi}{k+1}\Big)^{n-1}.$$

Note that since there are k possible initial letters for a staircase word and at most three possibilities thereafter for each subsequent letter, the number of staircase words is bounded above by $k3^{k-1}$. This upper bound becomes increasingly more accurate as k grows larger (except for a factor of $8\pi^2$ below), since as $k\to\infty$,

$$1+2\cos\frac{\pi}{k+1} = 3 - \frac{\pi^2}{k^2} + \frac{2\pi^2}{k^3} + O(k^{-4})$$

and

$$\frac{2}{k+1}\cot^2\frac{\pi}{2(k+1)} = \frac{8k}{\pi^2} + \frac{8}{\pi^2} + O\Big(\frac{1}{k}\Big).$$

3. STAIRCASE-CYCLIC WORDS

In this section we find an explicit formula for the generating function $scw_k(x) = \sum_{n\geq 0} scw_{n,k}x^n$.

Denote by $scw_k(x \mid i_1\cdots i_s \mid j)$ the generating function for the number of staircase-cyclic words $\sigma = \sigma_1\cdots\sigma_n$ of length n over $[k]$ such that $\sigma_1\cdots\sigma_s = i_1\cdots i_s$ and $\sigma_n = j$. We define

$$scw_k(x, v \mid i_1\cdots i_s) = \sum_{j=1}^{k} scw_k(x \mid i_1\cdots i_s \mid j)v^j.$$

Lemma 3.1. *For all $i = 1, 2, \ldots, k$,*

$$\begin{aligned}scw_k(x, v \mid i) &= (v^{i-1}[\![i > 1]\!] + v^i + v^{i+1}[\![k > i]\!])x^2 \\ &\quad + x(scw_k(x, v \mid i-1) + scw_k(x, v \mid i) + scw_k(x, v \mid i+1),\end{aligned}$$

where $scw_k(x, v \mid j) = 0$ for $j \notin [k]$ and $[\![P]\!] = 1$ if the condition P holds, and $[\![P]\!] = 0$ otherwise.

Proof. Let σ be any staircase-cyclic word containing at least two letters. If σ contains exactly two letters and $\sigma_1 = i$, then $j = i-1, i, i+1$, which gives the contribution $(v^{i-1}[\![i > 1]\!] + v^i + v^{i+1}[\![k > i]\!])x^2$. Otherwise, the second letter of $\sigma = i\sigma_2\cdots\sigma_{n-1}j$ is either $i-1, i$ or $i+1$. Hence, in terms of generating functions, we have

$$\begin{aligned}scw_k(x, v \mid i) &= (v^{i-1}[\![i > 1]\!] + v^i + v^{i+1}[\![k > i]\!])x^2 \\ &\quad + \big(scw_k(x, v \mid i-1) + scw_k(x, v \mid i) + scw_k(x, v \mid i+1)\big)x,\end{aligned} \tag{3.1}$$

for all $1 \leq i \leq k$, which completes the proof. □

Restating Lemma 3.1 as a matrix system we have

$$\mathbf{A}\begin{pmatrix} scw_k(x, v \mid 1) \\ scw_k(x, v \mid 2) \\ \vdots \\ scw_k(x, v \mid k-1) \\ scw_k(x, v \mid k) \end{pmatrix} = \begin{pmatrix} v + v^2 \\ v + v^2 + v^3 \\ \vdots \\ v^{k-2} + v^{k-1} + v^k \\ v^{k-1} + v^k \end{pmatrix} x^2, \tag{3.2}$$

where $\mathbf{A}$ is the tridiagonal matrix already defined in the previous section.

Theorem 3.2. *The generating function for the number of staircase-cyclic words of length n over an alphabet of k letters is given by*

$$scw_k(x) = 1 + \frac{kx(1+3x)}{(1+x)(1-3x)} - \frac{2(k+1)x}{(1+x)(1-3x)} \frac{U_{k-1}\left(\frac{1-x}{2x}\right)}{U_k\left(\frac{1-x}{2x}\right)}.$$

Proof. Equation (3.2) gives that

$$\begin{pmatrix} scw_k(x,v \mid 1) \\ scw_k(x,v \mid 2) \\ \vdots \\ scw_k(x,v \mid k-1) \\ scw_k(x,v \mid k) \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} v+v^2 \\ v+v^2+v^3 \\ \vdots \\ v^{k-2}+v^{k-1}+v^k \\ v^{k-1}+v^k \end{pmatrix} x^2,$$

where $\mathbf{A}^{-1}$ is defined in (2.2)

Fix $t = \dfrac{1-x}{2x}$ and i, where $i = 1, 2, \ldots, k$. By comparing the coefficients of v^j in the i-th row in the above matrix equation we obtain, for $i \neq 1, k$,

$$\sum_{j=1}^{k} scw_k(x \mid i \mid j) = x^2(\mathbf{A}^{-1}_{i(i-2)} + 2\mathbf{A}^{-1}_{i(i-1)} + 3\mathbf{A}^{-1}_{ii} + 2\mathbf{A}^{-1}_{i(i+1)} + \mathbf{A}^{-1}_{i(i+2)}),$$

and for $i = 1, k$ we have

$$\sum_{j=1}^{2} scw_k(x \mid 1 \mid j) = x^2(2\mathbf{A}^{-1}_{11} + 2\mathbf{A}^{-1}_{12} + \mathbf{A}^{-1}_{13}),$$

$$\sum_{j=k-1}^{k} scw_k(x \mid k \mid j) = x^2(2\mathbf{A}^{-1}_{kk} + 2\mathbf{A}^{-1}_{k(k-1)} + \mathbf{A}^{-1}_{k(k-2)}).$$

Note that $scw_k(x \mid i \mid j) = 0$ for $|j-i| > 1$. Thus the generating function $scw_k(x)$ is given by

$$1 + kx - x^2(\mathbf{A}^{-1}_{11} + \mathbf{A}^{-1}_{kk}) + x^2 \sum_{i=1}^{k} (\mathbf{A}^{-1}_{i(i-2)} + 2\mathbf{A}^{-1}_{i(i-1)} + 3\mathbf{A}^{-1}_{ii} + 2\mathbf{A}^{-1}_{i(i+1)} + \mathbf{A}^{-1}_{i(i+2)}),$$

where 1 counts the empty words, and kx counts the words of length 1 in the set of words over $[k]$. Therefore, applying (2.2) we obtain

$$\begin{aligned} scw_k(x) = 1 + kx &+ \frac{2x(2U_{k-1}(t) + 2U_{k-2}(t) + U_{k-3}(t))}{U_k(t)} \\ &+ \frac{x}{U_k(t)} \sum_{i=2}^{k-1} \big(U_{i-3}(t) + 2U_{i-2}(t) + 3U_{i-1}(t)\big) U_{k-i}(t) \\ &+ U_{i-1}(t)(2U_{k-1-i}(t) + U_{k-2-i}(t)), \end{aligned}$$

which, by simple algebraic operations, is equivalent to

$$\begin{aligned} scw_k(x) = 1 + kx &+ \frac{2x(2U_{k-1}(t) + 2U_{k-2}(t) + U_{k-3}(t))}{U_k(t)} \\ &+ \frac{x}{U_k(t)} \sum_{i=0}^{k-3} \big(2U_{i-1}(t) + 4U_i(t) + 3U_{i+1}(t)\big) U_{k-2-i}(t) \end{aligned}$$

and can be simplified to

$$\begin{aligned}
scw_k(x) = 1 + kx + \frac{x}{U_k(t)} \Big[& 4U_{k-1}(t) + 4U_{k-2}(t) + 2U_{k-3}(t) \\
& + \frac{x^2}{(1+x)(1-3x)} \Big(3(k-2)U_{k+1}(t) + 4(k-2)U_k(t) - kU_{k-1}(t) \\
& - 4(k-1)U_{k-2}(t) - 2(k+1)U_{k-3}(t) - 4U_{k-4}(t) - 2U_{k-5}(t) \Big) \Big].
\end{aligned}$$

Using the recursion (1.1) for the Chebyshev polynomials several times we arrive at

$$scw_k(x) = 1 + \frac{kx(1+3x)}{(1+x)(1-3x)} - \frac{2(k+1)x}{(1+x)(1-3x)} \frac{U_{k-1}(t)}{U_k(t)},$$

as claimed. □

We now give an explicit formula for the number of staircase-cyclic words of length n over the alphabet $[k]$.

Theorem 3.3. *The number of staircase-cyclic words of length n over the alphabet $[k]$ is given by*

$$scw_{n,k} = \sum_{j=1}^{k} \left[1 + 2\cos\Big(\frac{j\pi}{k+1}\Big) \right]^n.$$

Proof. Fix k and $\theta_j = \frac{j\pi}{k+1}$. Then Lemma 2.3 implies that the coefficient of x^n in $\frac{2(k+1)x}{(1+x)(1-3x)} \frac{U_{k-1}(t)}{U_k(t)}$, with $t = \frac{1-x}{2x}$, is given by

$$q_{n,k} = [x^n] \frac{4x^2}{(1+x)(1-3x)} \sum_{j=1}^{k} \frac{\sin^2\theta_j}{1 - x(1+2\cos\theta_j)}.$$

Using the fact that

$$\frac{1}{(1+x)(1-3x)(1-x\omega)} = \frac{1}{4(1+\omega)(1+x)} + \frac{\omega^2}{(\omega-3)(1+\omega)(1-x\omega)} - \frac{9}{4(\omega-3)(1-3x)},$$

and (2.7), we obtain that

$$\begin{aligned}
q_{n,k} &= \sum_{j=1}^{k} \sin^2\theta_j \left[\frac{(-1)^n}{4\cos^2(\theta_j/2)} - \frac{(1+2\cos\theta_j)^n}{\sin^2\theta_j} + \frac{3^n}{4\sin^2(\theta_j/2)} \right] \\
&= \sum_{j=1}^{k} \left[\sin^2(\theta_j/2)(-1)^n - (1+2\cos\theta_j)^n + \cos^2(\theta_j/2)3^n \right].
\end{aligned}$$

Using the identities $\sum_{j=1}^{k}\cos^2(\theta_j/2)=\frac{k}{2}$ and $\sum_{j=1}^{k}\sin^2(\theta_j/2)=\frac{k}{2}$, we get that

$$q_{n,k}=\frac{k}{2}\big((-1)^n+3^n\big)-\sum_{j=1}^{k}(1+2\cos\theta_j)^n.$$

Hence, Theorem 3.2 states that the coefficient of x^n, $n\geq 1$, in the generating function $scw_k(x)$ is given by

$$scw_{n,k}=\frac{k}{2}\big(3^n+(-1)^n\big)-q_{n,k}=\sum_{j=1}^{k}(1+2\cos\theta_j)^n,$$

as claimed. □

Corollary 3.4. *Asymptotically, we have as $n\to\infty$,*

$$scw_{n,k}\sim\left[1+2\cos\Big(\frac{\pi}{k+1}\Big)\right]^n.$$

Note that, asymptotically, staircase and staircase-cyclic words have the same exponential growth order, just a different constant. More precisely we may deduce

Corollary 3.5. *The proportion of staircase words that are staircase-cyclic in $[k]^n$ tends to $\frac{1}{2}(k+1)\left(2\cos\left(\frac{\pi}{k+1}\right)+1\right)\tan^2\left(\frac{\pi}{2(k+1)}\right)$ as $n\to\infty$.*

We observe that for large k,

$$\frac{1}{2}(k+1)\left(2\cos\Big(\frac{\pi}{k+1}\Big)+1\right)\tan^2\Big(\frac{\pi}{2(k+1)}\Big)=\frac{3\pi^2}{8k}-\frac{3\pi^2}{8k^2}+O(k^{-3}).$$

4. STAIRCASE NECKLACES

Staircase necklaces were defined in the introduction of the paper. To count the number $sn_{n,k}$ of staircase necklaces of length n over an alphabet of k letters we consider equivalence classes of staircase-cyclic words up to rotation. From Theorem 3.2 we then obtain the following result by a direct application of Theorem 3.2 and [**12**, Exercise 7.112(a)].

Theorem 4.1. *Let $n\geq 1$. The number $sn_{n,k}$ of staircase-necklaces of length n over an alphabet of k letters is given by*

$$sn_{n,k}=\frac{1}{n}\sum_{i=1}^{k}\sum_{j|n}\phi(j)\left[1+2\cos\Big(\frac{i\pi}{k+1}\Big)\right]^{n/j}$$

where ϕ is Euler's totient function ($\phi(n)$ is the number of positive integers $\leq n$ that are relatively prime to n), and we write $j\mid n$ if j divides n.

We see from this that asymptotically as $n \to \infty$, $sn_{n,k} \sim scw_{n,k}$.

The following table (Table 2) is obtained from Theorem 4.1. Interestingly only the sequences corresponding to the base cases, $k = 1$ and $k = 2$, appear in the database [**8**].

n	0	1	2	3	4	5	6	7	8	9	10	11
$sn_{n,1}$	1	1	1	1	1	1	1	1	1	1	1	1
$sn_{n,2}$	1	2	3	4	6	8	14	20	36	60	108	188
$sn_{n,3}$	1	3	5	7	12	19	39	71	152	315	685	1479
$sn_{n,4}$	1	4	7	10	18	30	65	128	293	658	1544	3622
$sn_{n,5}$	1	5	9	13	24	41	91	185	435	1009	2445	5945
$sn_{n,6}$	1	6	11	16	30	52	117	242	577	1360	3347	8278
$sn_{n,7}$	1	7	13	19	36	63	143	299	719	1711	4249	10611

Table 2. Numbers of smooth necklaces $sn_{n,k}$.

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