Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 242–252. doi:10.2298/AADM0902242B

OSCILLATION AND BOUNDEDNESS OF SOLUTIONS TO FIRST AND SECOND ORDER FORCED FUNCTIONAL DYNAMIC EQUATIONS WITH MIXED NONLINEARITIES

Martin Bohner and Taher S. Hassan

We investigate the oscillation and boundedness of first and second order dynamic equations with mixed nonlinearities. Our results extend and improve known results for oscillation of first and second order dynamic equations that have been established by Agarwal and Bohner. Some examples are given to illustrate the main results.

1. INTRODUCTION

Not only does the new theory of so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$ (which has important applications in quantum theory [**25**]) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$, and the space of harmonic numbers $\mathbb{T} = \{H_n\}$. In this work, a knowledge and understanding of time scales and time scale notation is assumed; for an introduction to the calculus on time scales, see [**8**, **9**, **24**]. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations. We refer the reader to [**1**, **2**, **4**–**23**] and the references cited therein. To the best of our knowledge, there is very little known about the oscillatory behavior and boundedness of first order dynamic equations. Recently, Agarwal and Bohner [**2**] (see also [**3**]) considered the first

2000 Mathematics Subject Classification. 39A10.
Keywords and Phrases. Dynamic equation, time scales, oscillations, mixed nonlinear, non-homogeneous

order dynamic equations

$$\begin{aligned} x^{\Delta}(t) - q(t)x^{\sigma}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} &= f(t), \\ x^{\Delta}(t) + q(t)x^{\sigma}(t) + q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \\ x^{\Delta}(t) + q(t)x^{\sigma}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} + q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \\ x^{\Delta}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} - q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \end{aligned}$$

and the second order dynamic equations

$$\begin{aligned} (r(x^{\Delta})^{\lambda})^{\Delta}(t) - q(t)x^{\sigma}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} &= f(t), \\ (r(x^{\Delta})^{\lambda})^{\Delta}(t) + q(t)x^{\sigma}(t) + q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \\ (r(x^{\Delta})^{\lambda})^{\Delta}(t) + q(t)x^{\sigma}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} + q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \\ (r(x^{\Delta})^{\lambda})^{\Delta}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} - q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} &= f(t), \end{aligned}$$

where λ, α_1 and β_1 are ratios of odd positive integers with $\alpha_1 > 1$ and $0 < \beta_1 < 1$ and q, q_1 and q_2 are positive rd-continuous functions on $\mathbb{T}$.

The purpose of this paper is to extend the oscillation and boundedness criteria to first order dynamic equations of the form

$$x^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_1(t)x^{\alpha}(h(t)) = f(t) \tag{1.1}$$

and

$$x^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_1(t)x^{\alpha}(h(t)) + q_2(t)x^{\beta}(h(t)) = f(t), \tag{1.2}$$

and we also examine second order dynamic equations of the form

$$(r(x^{\Delta})^{\lambda})^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_1(t)x^{\alpha}(h(t)) = f(t) \tag{1.3}$$

and

$$(r(x^{\Delta})^{\lambda})^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_1(t)x^{\alpha}(h(t)) + q_2(t)x^{\beta}(h(t)) = f(t), \tag{1.4}$$

where λ, γ, α and β are ratios of odd positive integers with $\alpha > \gamma > 0$ and $\beta > \gamma > 0$, p is an rd-continuous function on $\mathbb{T}$, and r, q_1 and q_2 are positive rd-continuous functions on $\mathbb{T}$. We assume that the function $h : \mathbb{T} \to \mathbb{T}$ satisfies $\lim_{t\to\infty} h(t) = \infty$, and hence our results will improve and extend results in [**2**].

Throughout this paper, we let

$$\begin{aligned} d_+ &:= \max\{0, d\}, \quad d_- := \max\{0, -d\}, \\ g_1 &:= (\alpha-\gamma)\alpha^{\alpha/(\gamma-\alpha)}\gamma^{\gamma/(\alpha-\gamma)}p_-^{\alpha/(\alpha-\gamma)}q_1^{\gamma/(\gamma-\alpha)} \end{aligned}$$

and, with $\delta > 0$,

$$\begin{aligned} g_2 := {} & (\alpha-\gamma)\alpha^{\alpha/(\gamma-\alpha)}\gamma^{\gamma/(\alpha-\gamma)}\delta^{\alpha/(\alpha-\gamma)}p_+^{\alpha/(\alpha-\gamma)}q_1^{\gamma/(\gamma-\alpha)} \\ & +(\beta-\gamma)\beta^{\beta/(\gamma-\beta)}\gamma^{\gamma/(\beta-\gamma)}(1+\delta)^{\beta/(\beta-\gamma)}p_-^{\beta/(\beta-\gamma)}q_2^{\gamma/(\gamma-\beta)}. \end{aligned}$$

Before stating our main results, we begin with the following lemma which will play an important rôle in the proof of our main results.

Lemma 1.1 *If a and b are nonnegative, then*

$$\gamma a^{\alpha} - \alpha a^{\gamma} b^{\alpha-\gamma} + (\alpha-\gamma) b^{\alpha} \geq 0 \quad \text{for all} \quad \alpha \geq \gamma > 0.$$

Proof. If $\alpha > \gamma > 0$, we let $f(u) = \gamma u^{\alpha} - \alpha b^{\alpha-\gamma} u^{\gamma}$. It is easy to see that f obtains its minimum at b and that $f(b) = -(\alpha-\gamma) b^{\alpha}$. □

2. FIRST ORDER DYNAMIC EQUATIONS

In this section, we give oscillation criteria for solutions to first order dynamic equations of the form (1.1) and (1.2).

Theorem 2.1. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} (f(\tau) + g_1(\tau))\,\Delta\tau = -\infty \text{ and } \limsup_{t\to\infty} \int_{t_0}^{t} (f(\tau) - g_1(\tau))\,\Delta\tau = \infty, \tag{2.1}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (2.1). By way of contradiction, assume that (1.1) is not oscillatory so that it has at least one eventually positive solution or at least one eventually negative solution. First we assume that x satisfies (1.1) and is eventually positive. Hence there exists $t_1 \in \mathbb{T}$, $t_1 \geq t_0$, such that $x(t) > 0$ and $x\big(h(t)\big) > 0$ for $t_1 \geq t_0$. From (1.1), we obtain for $t \geq t_1$

$$\begin{aligned} x^{\Delta}(t) &= f(t) - p(t) x^{\gamma}(h(t)) - q_1(t) x^{\alpha}(h(t)) \\ &\leq f(t) + p_-(t) x^{\gamma}(h(t)) - q_1(t) x^{\alpha}(h(t)). \end{aligned} \tag{2.2}$$

Define nonnegative functions a and b by

$$a(t) := \gamma^{-1/\alpha} q_1^{1/\alpha}(t) x(h(t)), \quad b(t) := \alpha^{1/(\gamma-\alpha)} \gamma^{\gamma/(\alpha(\alpha-\gamma))} p_-^{1/(\alpha-\gamma)}(t) q_1^{\gamma/(\alpha(\gamma-\alpha))}(t).$$

The last two terms in (2.2) can be written as

$$\alpha \left(\gamma^{-\gamma/\alpha} q_1^{\gamma/\alpha}(t) x^{\gamma}(h(t))\right) \left(\alpha^{-1} \gamma^{\gamma/\alpha} p_-(t) q_1^{-\gamma/\alpha}(t)\right) - \gamma \left(\gamma^{-1} q_1(t) x^{\alpha}(h(t))\right),$$

which is equal to

$$\alpha a^{\gamma}(t) b^{\alpha-\gamma}(t) - \gamma a^{\alpha}(t) \leq (\alpha-\gamma) b^{\alpha}(t) = g_1(t)$$

for $t \geq t_1$, where we have used Lemma 1.1. From this last inequality and (2.2), we get

$$x^{\Delta}(t) \leq f(t) + g_1(t),$$

and thus

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(\tau)\Delta\tau \le x(t_1) + \int_{t_1}^{t} (f(\tau) + g_1(\tau))\,\Delta\tau$$
$$= c + \int_{t_0}^{t} (f(\tau) + g_1(\tau))\,\Delta\tau,$$

where

$$c := x(t_1) - \int_{t_0}^{t_1} (f(\tau) + g_1(\tau))\,\Delta\tau.$$

Employing the first condition in (2.1), we find

$$0 \le \liminf_{t\to\infty} x(t) \le -\infty,$$

which is a contradiction. Next, we assume that x satisfies (1.1) and is eventually negative. If we put $y = -x$, then y is an eventually positive solution of (1.1), where f is replaced by $-f$. This case therefore leads to a contradiction as in the first case, provided

$$\liminf_{t\to\infty} \int_{t_0}^{t} (-f(\tau) + g_1(\tau))\,\Delta\tau = -\infty,$$

which indeed holds due to the second condition in (2.1). □

Corollary 2.1. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = -\infty \quad \textit{and} \quad \limsup_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \infty \tag{2.3}$$

and

$$\int_{t_0}^{\infty} p_-^{\alpha/(\alpha-\gamma)}(\tau) q_1^{\gamma/(\gamma-\alpha)}(\tau)\Delta\tau < \infty, \tag{2.4}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. As (2.3) and (2.4) imply (2.1), the claim follows from Theorem 2.1. □

Example 2.1. Let $\mathbb{T}$ be a time scale satisfying

$$\mu(t) := \sigma(t) - t \neq 0 \quad \text{for all} \quad t \in \mathbb{T}, \quad t_0 \in \mathbb{T}, \quad t_0 > 0, \tag{2.5}$$

and consider on $[t_0, \infty)_{\mathbb{T}}$ the equation

$$x^{\Delta}(t) + p(t)x^{1/3}(h(t)) + \frac{1}{t\,(\sigma(t))^6}\, x^{5/3}(h(t)) = \frac{t + \sigma(t)}{\mu(t)}\, e_{-2/\mu}(\sigma(t), t_0), \tag{2.6}$$

where, for any positive rd-continuous function ϕ on $\mathbb{T}$,

$$p(t) := \begin{cases} \dfrac{-1}{t\,(\sigma(t))^2} & \text{for} \quad t \in [2n, 2n+1)_{\mathbb{T}} \\ \phi(t) & \text{for} \quad t \in [2n+1, 2n+2)_{\mathbb{T}} \end{cases}$$

with $n \in \mathbb{N}$. As in [**2**, Example 2.1], we get

$$F^{\Delta}(t) = \frac{t+\sigma(t)}{\mu(t)}\, e_{-2/\mu}(\sigma(t),t_0) =: f(t) \quad \text{for} \quad F(t) := te_{-2/\mu}(t,t_0)$$

and

$$e_{-2/\mu}(\sigma(t),t_0) = -e_{-2/\mu}(t,t_0),$$

which implies

$$\liminf_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \liminf_{t\to\infty} \big(te_{-2/\mu}(t,t_0) - t_0\big) = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \limsup_{t\to\infty} \big(te_{-2/\mu}(t,t_0) - t_0\big) = \infty.$$

Moreover,

$$g_1(t) = \frac{4}{5\sqrt[4]{5}} \left(\frac{1}{t(\sigma(t))^2}\right)^{5/4} \left(\frac{1}{t(\sigma(t))^6}\right)^{-1/4} = \frac{4}{5\sqrt[4]{5}}\, \frac{1}{t\sigma(t)} \quad \text{for} \quad t \in [2n, 2n+1)_{\mathbb{T}}$$

and

$$g_1(t) = 0 \quad \text{for} \quad t \in [2n+1, 2n+2)_{\mathbb{T}},$$

which yields

$$\int_{2}^{t} g_1(\tau)\Delta\tau = \sum_{k=1}^{n} \frac{4}{5\sqrt[4]{5}} \int_{2k}^{2k+1} \frac{\Delta\tau}{\tau\sigma(\tau)} = \sum_{k=1}^{n} \frac{4}{5\sqrt[4]{5}}\, \frac{1}{2k(2k+1)} < \infty \quad \text{as} \quad n \to \infty.$$

Then by Corollary 2.1, each solution of (2.6) is oscillatory.

Theorem 2.2. *If*

$$\text{(2.7)}\quad \liminf_{t\to\infty} \int_{t_0}^{t} \left(f(\tau) + g_2(\tau)\right)\Delta\tau = -\infty \ \textit{and}\ \limsup_{t\to\infty} \int_{t_0}^{t} \left(f(\tau) - g_2(\tau)\right)\Delta\tau = \infty,$$

then every solution of equation (1.2) *is oscillatory.*

Proof. We proceed exactly as in the proof of Theorem 2.1. Let $\delta > 0$. From (1.2), we obtain, for $t \geq t_1$

$$\begin{aligned} x^{\Delta}(t) &= f(t) - p(t)x^{\gamma}(h(t)) - q_1(t)x^{\alpha}(h(t)) - q_2(t)x^{\beta}(h(t)) \\ &= f(t) + \delta p(t)x^{\gamma}(h(t)) - q_1(t)x^{\alpha}(h(t)) \\ &\qquad\qquad - (1+\delta)p(t)x^{\gamma}(h(t)) - q_2(t)x^{\beta}(h(t)) \\ &\leq f(t) + \delta p_+(t)x^{\gamma}(h(t)) - q_1(t)x^{\alpha}(h(t)) \\ &\qquad\qquad + (1+\delta)p_-(t)x^{\gamma}(h(t)) - q_2(t)x^{\beta}(h(t)). \end{aligned}$$

Define $a_1, a_2 \geq 0$ and $b_1, b_2 \geq 0$ by

$$\begin{aligned} a_1(t) &:= \gamma^{-1/\alpha} q_1^{1/\alpha}(t)x(h(t)), \\ b_1(t) &:= \alpha^{1/(\gamma-\alpha)} \gamma^{\gamma/(\alpha(\alpha-\gamma))} \delta^{1/(\alpha-\gamma)} p_+^{1/(\alpha-\gamma)}(t) q_1^{\gamma/(\alpha(\gamma-\alpha))}(t) \end{aligned}$$

and

$$a_2(t) := \gamma^{-1/\beta} q_2^{1/\beta}(t) x(h(t)),$$
$$b_2(t) := \alpha^{1/(\gamma-\beta)} \gamma^{\gamma/(\beta(\beta-\gamma))} (1+\delta)^{1/(\beta-\gamma)} p_-^{1/(\beta-\gamma)}(t) q_2^{\gamma/(\beta(\gamma-\beta))}(t),$$

which implies by Lemma 1.1 that

$$\begin{aligned}\delta p_+(t)x^\gamma(h(t)) - q_1(t)x^\alpha(h(t)) &= \alpha \left(\gamma^{-1/\alpha} q_1^{1/\alpha}(t)x(h(t))\right)^\gamma \times \\ &\times \left(\alpha^{1/(\gamma-\alpha)}\gamma^{\gamma/(\alpha(\alpha-\gamma))}\delta^{1/(\alpha-\gamma)} p_+^{1/(\alpha-\gamma)}(t) q_1^{\gamma/(\alpha(\gamma-\alpha))}(t)\right)^{\alpha-\gamma} \\ &\quad - \gamma\left(\gamma^{-1/\alpha} q_1^{1/\alpha}(t)x(h(t))\right)^\alpha \\ &= \gamma a_1^\alpha(t) - \alpha a_1^\gamma(t) b_1^{\alpha-\gamma}(t) \le (\alpha-\gamma) b_1^\alpha(t)\end{aligned}$$

and

$$\begin{aligned}(1+\delta)p_-(t)x^\gamma(h(t)) - q_2(t)x^\beta(h(t)) &= \beta \left(\gamma^{-1/\beta} q_2^{1/\beta}(t)x(h(t))\right)^\gamma \times \\ &\times \left(\beta^{1/(\gamma-\beta)}\gamma^{\gamma/(\beta(\beta-\gamma))}(1+\delta)^{1/(\beta-\gamma)} p_-^{1/(\beta-\gamma)}(t) q_2^{\gamma/(\beta(\gamma-\beta))}(t)\right)^{\beta-\gamma} \\ &\quad - \gamma\left(\gamma^{-1/\beta} q_2^{1/\beta}(t)x(h(t))\right)^\beta \\ &= \gamma a_2^\beta(t) - \beta a_2^\gamma(t) b_2^{\beta-\gamma}(t) \le (\beta-\gamma) b_2^\beta.\end{aligned}$$

It follows that

$$x^\Delta(t) \le f(t) + g_2(t),$$

and the rest of the proof is line by line the same as the proof of Theorem 2.1 with g_1 replaced by g_2. □

Corollary 2.2. *If* (2.3) *holds and*

$$\int_{t_0}^\infty p_+^{\alpha/(\alpha-\gamma)}(\tau) q_1^{\gamma/(\gamma-\alpha)}(\tau)\Delta\tau < \infty \text{ and } \int_{t_0}^\infty p_-^{\beta/(\beta-\gamma)}(\tau) q_2^{\gamma/(\gamma-\beta)}(\tau)\Delta\tau < \infty, \tag{2.8}$$

then every solution of equation (1.2) *is oscillatory.*

Proof. As (2.3) and (2.8) imply (2.7), the claim follows from Theorem 2.2. □

Remark 2.1.

(a) Let $\gamma = 1$, $\alpha > 1$, $h(t) = t$, and p be a negative rd-continuous function on $\mathbb{T}$. Then Theorem 2.1 and Corollary 2.1 reduce to [**2**, Theorem 2.1 and Corollary 2.2].

(b) Let $0 < \gamma < 1$, $\alpha = 1$, $h(t) = t$, and p be a negative rd-continuous function on $\mathbb{T}$. Then Theorem 2.1 and Corollary 2.1 reduce to [**2**, Theorem 2.3 and Corollary 2.4].

(c) Let $0 < \gamma < 1$, $\alpha > 1$, $h(t) = t$, and p be a negative rd-continuous function on $\mathbb{T}$. Then Theorem 2.1 and Corollary 2.1 reduce to [**2**, Theorem 2.7 and Corollary 2.8].

(d) Let $0 < \gamma < 1$, $\alpha > 1$, $\beta = 1$, $h(t) = t$, and p be a negative rd-continuous function on $\mathbb{T}$. Then Theorem 2.2 and Corollary 2.2 reduce to [**2**, Theorem 2.7 and Corollary 2.8].

(e) If $h(t) \neq t$ or p is not negative, then the results presented above are new.

EXAMPLE 2.2. Let $\mathbb{T}$ be a time scale satisfying (2.5) and consider on $[t_0, \infty)_{\mathbb{T}}$ the equation

$$x^{\Delta}(t)+p(t)x^{1/3}(h(t))+\frac{1}{t(\sigma(t))^9}x^{7/5}(h(t))+\frac{1}{t\sigma(t)}x(h(t)) = \frac{t+\sigma(t)}{\mu(t)}e_{-2/\mu}(\sigma(t),t_0), \tag{2.9}$$

where

$$p(t) := \begin{cases} \dfrac{-1}{t\sigma(t)} & \text{for} \quad t \in [2n, 2n+1)_{\mathbb{T}} \\ \dfrac{1}{t(\sigma(t))^2} & \text{for} \quad t \in [2n+1, 2n+2)_{\mathbb{T}} \end{cases}$$

with $n \in \mathbb{N}$. As in Example 2.1, (2.3) is satisfied and with $\delta = 1/2$,

$$g_2(t) = \frac{6}{5}\sqrt[4]{\frac{3}{10}}\,\frac{1}{t\sigma(t)} \quad \text{for} \quad t \in [2n, 2n+1)_{\mathbb{T}}$$

and

$$g_2(t) = \frac{3}{7\sqrt[6]{14}}\,\frac{1}{t\sigma(t)} \quad \text{for} \quad t \in [2n+1, 2n+2)_{\mathbb{T}}$$

implies as in Example 2.1 that (2.8) is satisfied. By Corollary 2.2, each solution of (2.9) is oscillatory.

We conclude this section by mentioning that the same arguments as above may be used to establish criteria that guarantee that all nonoscillatory solutions of any of the equations (1.1) and (1.2) are bounded. This is summarized as follows.

Teorem 2.3. *If* (2.4) *holds and*

$$\int_{t_0}^{\infty} |f(\tau)|\,\Delta\tau < \infty, \tag{2.10}$$

then every nonoscillatory solution of equation (1.1) *is bounded.*

Theorem 2.4. *If* (2.8) *and* (2.10) *hold, then every nonoscillatory solution of equation* (1.2) *is bounded.*

3. SECOND ORDER DYNAMIC EQUATIONS

In this section, we give oscillation criteria for solutions to second order dynamic equations of the form (1.3) and (1.4).

Theorem 3.1. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left(\frac{c}{r(s)} + \frac{1}{r(s)}\int_{t_0}^{s} (f(\tau)+g_1(\tau))\,\Delta\tau\right)^{1/\lambda} \Delta s = -\infty \tag{3.1}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{c}{r(s)}+\frac{1}{r(s)}\int_{t_0}^{s}(f(\tau)-g_1(\tau))\,\Delta\tau\right)^{1/\lambda}=\infty \tag{3.2}$$

for all $c\in\mathbb{R}$, then every solution of equation (1.3) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we assume that x is an eventually positive solution of (1.3). Hence there exists $t_1\in\mathbb{T}$, $t_1\geq t_0$, such that $x(t)>0$ and $x\big(h(t)\big)>0$ for $t_1\geq t_0$. As in the proof of Theorem 2.1, by Lemma 1.1, we get for $t\geq t_1$

$$(r(x^{\Delta})^{\lambda})^{\Delta}(t)=f(t)-p(t)x^{\gamma}(h(t))-q_1(t)x^{\alpha}(h(t))\leq f(t)+g_1(t).$$

Through integration, we obtain for $t\geq t_1$

$$r(t)(x^{\Delta}(t))^{\lambda}\leq r(t_1)(x^{\Delta}(t_1))^{\lambda}+\int_{t_1}^{t}(f(\tau)+g_1(\tau))\,\Delta\tau=c+\int_{t_0}^{t}(f(\tau)+g_1(\tau))\,\Delta\tau,$$

where

$$c:=r(t_1)(x^{\Delta}(t_1))^{\lambda}-\int_{t_0}^{t_1}(f(\tau)+g_1(\tau))\,\Delta\tau.$$

Therefore, for $t\geq t_1$

$$\begin{aligned}x(t)&\leq x(t_1)+\int_{t_1}^{t}\left(\frac{c}{r(s)}+\frac{1}{r(s)}\int_{t_0}^{s}(f(\tau)+g_1(\tau))\,\Delta\tau\right)^{1/\lambda}\Delta s\\&=c_*+\int_{t_0}^{t}\left(\frac{c}{r(s)}+\frac{1}{r(s)}\int_{t_0}^{s}(f(\tau)+g_1(\tau))\,\Delta\tau\right)^{1/\lambda}\Delta s,\end{aligned}$$

where

$$c_*:=x(t_1)-\int_{t_0}^{t_1}\left(\frac{c}{r(s)}+\frac{1}{r(s)}\int_{t_0}^{s}(f(\tau)+g_1(\tau))\,\Delta\tau\right)^{1/\lambda}\Delta s.$$

Employing condition (3.1), we find

$$0\leq\liminf_{t\to\infty}x(t)\leq-\infty,$$

which is a contradiction. The case of an eventually negative solution of (1.3) can be dealt with as in the proof of Theorem 2.1, this time employing (3.2). □

Corollary 3.1. *If* (2.3) *and* (2.4) *hold and*

$$\liminf_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_{t_0}^{s}f(\tau)\Delta\tau\right)^{1/\lambda}\Delta s=-\infty \tag{3.3}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_{t_0}^{s}f(\tau)\Delta\tau\right)^{1/\lambda}\Delta s=\infty, \tag{3.4}$$

then every solution of equation (1.3) *is oscillatory.*

Proof. We show that (2.3), (2.4), (3.3) and (3.4) imply (3.1) and (3.2) so that the claim follows from Theorem 3.1. By (2.3) and (2.4), there exists $T \geq t_0$ such that

$$\int_{t_0}^{T} f(\tau)\Delta\tau \leq -c - \int_{t_0}^{\infty} g_1(\tau)\Delta\tau \quad \text{for all} \quad c \in \mathbb{R}.$$

Then

$$\begin{aligned}
&\int_{t_0}^{t} \left(\frac{c}{r(s)} + \frac{1}{r(s)} \int_{t_0}^{s} \left(f(\tau) + g_1(\tau)\right)\Delta\tau \right)^{1/\lambda} \Delta s \\
&\leq \int_{t_0}^{t} \left(\frac{1}{r(s)} \left[c + \int_{t_0}^{\infty} g_1(\tau)\Delta\tau + \int_{t_0}^{T} f(\tau)\Delta\tau \right] + \frac{1}{r(s)} \int_{T}^{s} f(\tau)\Delta\tau \right)^{1/\lambda} \Delta s \\
&\leq \int_{t_0}^{t} \left(\frac{1}{r(s)} \int_{T}^{s} f(\tau)\Delta\tau \right)^{1/\lambda} \Delta s = \check{c} + \int_{T}^{t} \left(\frac{1}{r(s)} \int_{T}^{s} f(\tau)\Delta\tau \right)^{1/\lambda} \Delta s,
\end{aligned}$$

where

$$\check{c} := \int_{t_0}^{T} \left(\frac{1}{r(s)} \int_{T}^{s} f(\tau)\Delta\tau \right)^{1/\lambda} \Delta s,$$

so that condition (3.1) follows. By a similar argument, condition (3.2) holds as well. □

Remark 3.1. Let $\gamma = 1$, $\alpha > 1$, $h(t) = t$, and p be a negative rd-continuous function on $\mathbb{T}$. Then Theorem 3.1 and Corollary 3.2 reduce to [**2**, Corollaries 3.2 and 3.3].

As in Theorems 2.2 and 3.1 and Corollary 3.1, we get the following oscillation criteria for equation (1.4).

Corollary 3.2. *If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left(\frac{c}{r(s)} + \frac{1}{r(s)} \int_{t_0}^{s} \left(f(\tau) + g_2(\tau)\right)\Delta\tau \right)^{1/\lambda} \Delta s = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{c}{r(s)} + \frac{1}{r(s)} \int_{t_0}^{s} \left(f(\tau) - g_2(\tau)\right)\Delta\tau \right)^{1/\lambda} = \infty$$

for all $c \in \mathbb{R}$, *then every solution of equation* (1.4) *is oscillatory.*

Corollary 3.3. *If* (2.3), (2.8), (3.3) *and* (3.4) *hold, then every solution of equation* (1.4) *is oscillatory.*

Example 3.1. Let $\mathbb{T}$ be a time scale satisfying (2.5) and consider on $[t_0, \infty)_{\mathbb{T}}$ the equation

$$\left(\frac{t^2(\sigma(t))^2}{t + \sigma(t)} x^{\Delta} \right)^{\Delta} + p(t)x^{1/3}(h(t)) + \frac{1}{t(\sigma(t))^9} x^{7/5}(h(t)) + \frac{1}{t\sigma(t)} x(h(t)) = G^{\Delta}(t), \tag{3.5}$$

where

$$G(t) = \frac{t^2(\sigma(t))^2}{\mu(t)}\, e_{-2/\mu}(\sigma(t), t_0)$$

and p is as in Example 2.2. As in [**2**, Example 3.1], (3.3) and (3.4) are satisfied and also, as in Example 2.2, (2.8) is satisfied. By Corollary 3.3, each solution (3.5) is oscillatory.

Remark 3.2. The recent results due to Agarwal and Bohner [**2**] do not apply to equations (2.6), (2.9) and (3.5).

Also, as in Theorems 2.3 and 2.4, we can obtain results about the boundedness of nonoscillatory solutions of equations (1.3) and (1.4).

Corollary 3.4. *If* (2.4) *holds and*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(s)} + \frac{1}{r(s)} \int_{t_0}^{s} (|f(\tau)| + g_1(\tau))\, \Delta\tau \right)^{1/\lambda} \Delta s < \infty, \tag{3.6}$$

then every nonoscillatory solution of equation (1.3) *is bounded.*

Corollary 3.5. *If* (2.8) *and* (3.6) *hold, then every nonoscillatory solution of equation* (1.4) *is bounded.*

REFERENCES

1. R. Agarwal, M. Bohner, W.-T. Li: *Nonoscillation and Oscillation Theory for Functional Differential Equations.* Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., 2004.
2. R. P. Agarwal, M. Bohner: *Oscillation and boundedness of solutions to first and second order forced dynamic equations with mixed nonlinearities.* Aust. J. Math. Anal. Appl., **5** (1):1–12, 2008. Article 2.
3. R. P. Agarwal, M. Bohner, W. S. Cheung, S. R. Grace: *Oscillation criteria for first and second order forced difference equations with mixed nonlinearities.* Math. Comput. Modelling, **45** (7-8) (2007), 965–973.
4. R. P. Agarwal, M. Bohner, S. R. Grace: *Oscillation criteria for first-order forced nonlinear dynamic equations.* Can. Appl. Math. Q., **15** (3) (2008), 223–233.
5. R. P. Agarwal, M. Bohner, S. R. Grace: *On the oscillation of second-order half-linear dynamic equations.* J. Difference Equ. Appl., **15** (5) (2009), 451–460.
6. R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan: *Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations.* Commun. Nonlinear Sci. Numer. Simul., **14** (8) (2009), 3463–3471.
7. R. P. Agarwal, M. Bohner, S. H. Saker: *Oscillation of second order delay dynamic equations.* Can. Appl. Math. Q., **13** (1) (2005), 1–17.
8. M. Bohner, A. Peterson: *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhäuser, Boston, 2001.
9. M. Bohner, A. Peterson: *Advances in Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2003.
10. M. Bohner, S. H. Saker: *Oscillation criteria for perturbed nonlinear dynamic equations.* Math. Comput. Modelling, **40** (3-4) (2004), 249–260.

11. M. Bohner, S. H. Saker: *Oscillation of second order nonlinear dynamic equations on time scales.* Rocky Mountain J. Math., **34** (4) (2004), 1239–1254.
12. M. Bohner, C. C. Tisdell: *Oscillation and nonoscillation of forced second order dynamic equations.* Pacific J. Math., **230** (1) (2007), 59–71.
13. L. Erbe, T. S. Hassan, A. Peterson: *Oscillation criteria for nonlinear damped dynamic equations on time scales.* Appl. Math. Comput., **203** (1) (2008), 343–357.
14. L. Erbe, T. S. Hassan, A. Peterson: *Forced oscillation of second order functional differential equations with mixed nonlinearities.* Acta Math. Sci., (2009). To appear.
15. L. Erbe, T. S. Hassan, A. Peterson: *Oscillation criteria for forced second order functional dynamic equations on time scales with mixed nonlinearities.* Adv. Dyn. Syst. Appl, (2009). Submitted.
16. L. Erbe, T. S. Hassan, A. Peterson: *Oscillation criteria for nonlinear functional neutral dynamic equations on time scales.* J. Difference Equ. Appl., (2009). To appear.
17. L. Erbe, T. S. Hassan, A. Peterson: *Oscillation of third order nonlinear functional dynamic equations on time scales.* Differential Equations Dynam. Systems, (2009). To appear.
18. L. Erbe, T. S. Hassan, A. Peterson, S. Saker: *Interval oscillation criteria for forced second order nonlinear delay dynamic equations with oscillatory potential.* Dynam. Contin. Discrete Impuls. Systems, (2009). To appear.
19. L. Erbe, T. S. Hassan, A. Peterson, S. Saker: *Oscillation criteria for half-linear delay dynamic equations on time scales.* Nonlinear Dyn. Syst. Theory, **9** (2009), 51–68.
20. L. Erbe, T. S. Hassan, A. Peterson, S. Saker: *Oscillation criteria for sublinear half-linear delay dynamic equations on time scales.* Int. J. Difference Equ., (2009). To appear.
21. L. Erbe, A. Peterson, S. H. Saker: *Oscillation criteria for second-order nonlinear dynamic equations on time scales.* J. London Math. Soc., **67** (3) (2003), 701–714.
22. T. S. Hassan: *Oscillation criteria for half-linear dynamic equations on time scales.* J. Math. Anal. Appl., **345** (1) (2008), 176–185.
23. T. S. Hassan: *Oscillation of third order nonlinear delay dynamic equations on time scales.* Math. Comput. Modelling, **49** (2009), 1573–1586.
24. S. Hilger: *Analysis on measure chains – a unified approach to continuous and discrete calculus.* Results Math., **18** (1990), 18–56.
25. V. Kac, P. Cheung: *Quantum Calculus.* Universitext. Springer-Verlag, New York, 2002.

(Received February 15, 2009)
(Revised May 25, 2009)

Missouri University of Science and Technology
Department of Mathematics and Statistics,
Department of Economics,
Rolla, Missouri 65401,
USA
E–mail: bohner@mst.edu
URL: http://web.mst.edu/~bohner

Mansoura University, Department of Mathematics,
Faculty of Science, Mansoura University,
Mansoura, 35516,
Egypt
E–mail tshassan@mans.edu.eg
URL: http://www.mans.edu.eg/pcvs/30140