Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 236–241. doi:10.2298/AADM0902236A

BANACH CONTRACTION PRINCIPLE ON CONE RECTANGULAR METRIC SPACES

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We introduce the notion of cone rectangular metric space and prove Banach contraction mapping principle in cone rectangular metric space setting. Our result extends recent known results.

1. INTRODUCTION AND PRELIMINARIES

Guang and Zhang [**2**] recently introduced the concept of cone metric space and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors [**1**, **3**, **4**, **5**] studied the existence of fixed points and common fixed points of mappings satisfying a contractive type condition on a normal cone metric space. In this paper we introduce cone rectangular metric spaces and prove Banach contraction mapping principle in a complete normal cone rectangular metric space.

A subset P of a real Banach space E is called a *cone* if it has following properties:

(i) P is nonempty closed and $P \neq \{\mathbf{0}\}$;

(ii) $0 \leq a, b \in R$ and $x,\ y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\mathbf{0}\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$,

2000 Mathematics Subject Classification. 47H10, 54H25.
Keywords and Phrases. Fixed point, contraction mapping principle, cone rectangular metric space, cone metric space.

while $x \ll y$ will stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $\kappa \geq 1$ such that for all $x,\ y \in E$,

$$0 \leq x \leq y \ \Rightarrow \|x\| \leq \kappa \|y\| . \tag{1}$$

The least value of κ satisfying (1) is called the *normal constant* of P.

In the following we always suppose that E is a real BANACH space and P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P. For more details see [**2**, **3**, **4**].

Definition 1. [**2**] *Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \rightarrow E$, satisfies:*

(1) $\mathbf{0} \leq \rho(x, y)$, *for all* $x, y \in X$ *and* $\rho(x, y) = \mathbf{0}$ *if and only if* $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, *for all* $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

EXAMPLE 1. [**2**] Let $E = R^2, \rho : X \times X \rightarrow E$ such that

$$\rho(x, y) = (|x - y| , \beta |x - y|) , \ \beta \geq 0$$
$$P = \{(x, y) : x, y \geq 0\} \subset R^2.$$

Then(X, ρ) is a *cone metric space.*

Definition 2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$, satisfies:*

(1) $\mathbf{0} \leq d(x, y)$, *for all* $x, y \in X$ *and* $d(x, y) = \mathbf{0}$ *if and only if* $x = y$;

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$:

(3) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ *for all* $x, y, \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$ [*rectangular property*].

Then d is called a *cone rectangular metric* on X, and (X, d) is called a *cone rectangular metric space*. Let $\{x_n\}$ be a sequence in (X, d) and $x \in (X, d)$. If for every $c \in E$, with $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \rightarrow x$, as $n \rightarrow \infty$. If for every $c \in E$ with $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0,\ d(x_n, x_{n+m}) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in (X, d). If every CAUCHY sequence is convergent in (X, d), then (X, d) is called a *complete cone rectangular metric space.*

EXAMPLE 2. Let $X = \mathbb{N}$, $E = R^2$ and

$$P = \{(x, y) : x, y \geq 0\}.$$

Define $d : X \times X \rightarrow E$ as follow:

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y, \\ (3, 9) & \text{if } x \text{ and } y \text{ are in } \{1, 2\}\ x \neq y. \\ (1, 3) & \text{otherwise} \end{cases}$$

Now (X, d) is a cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$(3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 3) + (1, 3) = (2, 6)$$

as $(3, 9) - (2, 6) = (1, 3) \in P$.

2. MAIN RESULTS

First we present two theorems whose proofs are similar to Huang and Zhang [**2**, Lemmas 1 and 4]

Theorem 1. *Let (X, d) be a rectangular cone metric space and P be a normal cone with normal constant κ. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.*

Theorem 2. *Let (X, d) be a rectangular cone metric space, P be a normal cone with normal constant κ.Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.*

Theorem 3. *Let (X, d) be a cone rectangular metric space, P be a normal cone with normal constant κ and the mapping $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq \lambda\ d(x, y)$$

for all $x, y \in X$, where $0 \leq \lambda < 1$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. Define a sequence of points in X as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0,\ n = 0, 1, 2, \ldots.$$

We can suppose that x_0 is not a periodic point, in fact if $x_n = x_0$, then,

$$\begin{aligned} d(x_0, Tx_0) = d(x_n, Tx_n) &= d(T^n x_0, T^{n+1}x_0) \leq \lambda d(T^{n-1}x_0, T^n x_0) \\ &\leq \lambda^2 d(T^{n-2}x_0, T^{n-1}x_0) \leq \cdots \leq \lambda^n d(x_0, Tx_0). \end{aligned}$$

It follows that

$$[\lambda^n - 1]\ d(x_0, Tx_0) \in P.$$

It further implies that

$$\left[\frac{\lambda^n - 1}{1 - \lambda^n}\right]\ d(x_0, Tx_0) \in P.$$

Hence $-d(x_0, Tx_0) \in P$ and $d(x_0, Tx_0) = \mathbf{0}$, this means x_0 is a fixed point of T. Thus in this sequel of proof we can suppose that $x_m \neq x_n$ for all distinct $m, n \in \mathbb{N}$. Now by using rectangular property for all $y \in X$, we have,

$$\begin{aligned} d(y, T^4 y) &\leq d(y, Ty) + d(Ty, T^2 y) + d(T^2 y, T^4 y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, T^2 y). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y,T^6y) &\le d(y,Ty)+d(Ty,T^2y)+d(T^{\ 2}y,T^3y)+d(T^3y,T^4y)+d(T^4y,T^6y) \\ &\le d(y,Ty)+\lambda d(y,Ty)+\lambda^2 d(y,Ty)+\lambda^3 d(y,T^2y)+\lambda^4 d(y,T^2y) \\ &\le \sum_{i=0}^{3}\lambda^i d(y,Ty)+\lambda^4\ d(y,T^2y), \text{ for all } y\in X. \end{aligned}$$

Now by induction, we obtain for each $k=2,3,4,\dots,$

$$d(y,T^{2k}y)\le \sum_{i=0}^{2k-3}\lambda^i d(y,Ty)+\lambda^{2k-2}\ d(y,T^2y). \tag{2}$$

Moreover, for all $y\in X$,

$$\begin{aligned} d(y,T^5y) &\le d(y,Ty)+d(Ty,T^2y)+d(T^2y,T^3y)+d(T^3y,T^4y)+d(T^4y,T^5y) \\ &\le \sum_{i=0}^{4}\lambda^i d(y,Ty). \end{aligned}$$

By induction, for each $k=0,1,2,\dots$ we have,

$$d(y,T^{2k+1}y)\le \sum_{i=0}^{2k}\lambda^i d(y,Ty). \tag{3}$$

Using inequality (2), for $k=1,2,3,\dots$ we have,

$$\begin{aligned} d(T^nx_0,T^{n+2k}x_0) &\le \lambda^n d(x_0,T^{2k}x_0) \\ &\le \lambda^n\Bigg[\sum_{i=0}^{2k-3}\lambda^i\left(d(x_0,Tx_0)+d(x_0,T^2x_0)\right) \\ &\qquad\qquad +\lambda^{2k-2}\left(d(x_0,Tx_0)+d(x_0,T^2x_0)\right)\Bigg] \\ &\le \lambda^n\sum_{i=0}^{2k-2}\lambda^i\left[d(x_0,Tx_0)+d(x_0,T^2x_0)\right] \\ &\le \frac{\lambda^n(1-\lambda^{2k-1})}{1-\lambda}\left[d(x_0,Tx_0)+d(x_0,T^2x_0)\right] \\ &\le \frac{\lambda^n}{1-\lambda}\left[d(x_0,Tx_0)+d(x_0,T^2x_0)\right]. \end{aligned}$$

Similarly, for $k=0,1,2,\dots,$ inequality (3) implies that

$$\begin{aligned} d(T^nx_0,T^{n+2k+1}x_0) &\le \lambda^n d(x_0,T^{2k+1}x_0) \\ &\le \lambda^n\sum_{i=0}^{2k}\lambda^i d(x_0,Tx_0) \\ &\le \frac{\lambda^n}{1-\lambda}\left[d(x_0,Tx_0)+d(x_0,T^2x_0)\right]. \end{aligned}$$

Thus,

$$d(T^n x_0, T^{n+m} x_0) \leq \frac{\lambda^n}{1-\lambda} \left[d(x_0, Tx_0) + d(x_0, T^2 x_0) \right].$$

Since P is a normal cone with normal constant κ, therefore,

$$\left\| d(T^n x_0, T^{n+m} x_0) \right\| \leq \frac{\lambda^n}{1-\lambda} \kappa \left\| \left[d(x_0, Tx_0) + d(x_0, T^2 x_0) \right] \right\|.$$

Therefore, $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$. Now Lemma 2 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$. By Lemma 1, we have

$$\|d(T^{\,n} x_0, u)\| \to 0 \text{ as } n \to \infty.$$

Since $x_n \neq x_m$ for $n \neq m$, therefore by rectangular property, we have

$$\begin{aligned} d(Tu, u) &\leq \lambda d(u, T^{n-1} x_0) + d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, u) \\ &\leq \lambda d(u, T^{n-1} x_0) + \lambda^n d(x_0, Tx_0) + d(T^{n+1} x_0, u). \end{aligned}$$

Thus,

$$\|d(Tu, u)\| \leq \kappa \left[\lambda \left\| d(u, T^{\,n-1} x_0) \right\| + \lambda^n \left\| d(x_0, Tx_0) \right\| + \left\| d(T^{\,n+1} x_0, u) \right\| \right].$$

Letting $n \to \infty$, we have

$$\|d(u, Tu)\| = 0.$$

Hence $u = Tu$. Now we show that T have a unique fixed point. For this, assume that there exists another point v in X such that $v = Tv$. Now,

$$d(v, u) = d(Tv, Tu) \leq \lambda \, d(v, u).$$

Hence, $u = v$. □

Example. Let $X = \{1, 2, 3, 4\}$, $E = R^2$ and

$$P = \{(x, y) : x, y \geq 0\}$$

is a normal cone in E. Define $d : X \times X \to E$ as follow:

$$\begin{aligned} d(1,2) &= d(2,1) = (3,6) \\ d(2,3) &= d(3,2) = d(1,3) = d(3,1) = (1,2) \\ d(1,4) &= d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = (2,4). \end{aligned}$$

Then $(X, d\ )$ is a complete cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$(3,6) = d(1,2) > d(1,3) + d(3,2) = (1,2) + (1,2) = (2,4)$$

as $(3,6) - (2,4) = (1,2) \in P$. Now define a mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 3 & if \ x \neq 4, \\ 1 & if \ x = 4. \end{cases}$$

Note that

$$d(T(1),T(2)) = d(T(1),T(3)) = d(T(2),T(3)) = \mathbf{0}$$

and in all other cases

$$d(Tx,Ty) = (1,2),\ d(x,y) = (2,4).$$

Hence, for $\lambda = 1/2$, all conditions of Theorem 3 are satisfied and 3 is a unique fixed point of T.

Remark 1. In the example 3, results of Huang and Zhang [**2**] are not applicable to obtain the fixed point of the mapping T on X, since (X,d) is not a cone metric space.

Acknowledgement. The present version of the paper owes much to the precise and kind remarks of the learned referee.

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(Received October 13, 2008)
(Revised May 10, 2009)

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