Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **2** (2008), 158–174. doi:10.2298/AADM0802158G

MULTIPLICITY OF SOLUTIONS FOR SINGULAR SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL HARDY-SOBOLEV EXPONENTS

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We consider the semilinear elliptic problem with critical Hardy-Sobolev exponents and Dirichlet boundary condition. By using variational methods we obtain the existence and multiplicity of nontrivial solutions and improve the former results.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the following wide class of semilinear elliptic problems,

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = g(x,u) + \beta \dfrac{|u|^{2^*(s)-2}}{|x|^s} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is an open bounded domain with smooth boundary, $\beta > 0,\ 0 \in \Omega,\ 0 \leq s < 2,\ 2^*(s) := \dfrac{2(N-s)}{N-2}$ is the critical Hardy-Sobolev exponent and, when $s = 0, 2^*(0) = \dfrac{2N}{N-2}$ is the critical Sobolev exponent, $0 \leq \mu < \overline{\mu} := \dfrac{(N-2)^2}{4}$.

In [**1**] A. Ferrero and F. Gazzola investigated the existence of nontrivial solutions for problem (1.1) with $\beta = 1, s = 0$. In [**2**] D. S. Kang and S. J. Peng

2000 Mathematics Subject Classification. 35J20,47J30.
Keywords and Phrases. Critical Hardy-Sobolev exponents, Hardy potential, variational method.
The project is supported by Natural Science Basic Research Plan in Shaanxi Province of China, Program No.2006A09.

dealt with (1.1) with $\beta = 1$ and $g(x,t) = \lambda|t|^{q-2}t$ and obtained the existence of one positive solution for suitable q and λ. They also proved in [**3**] that (1.1) has one nontrivial solution for $g(x,t) = \lambda t\,(\lambda > 0)$ and in [**9**] that (1.1) has one pair of sign-changing solutions for $g(x,t) = \lambda t\,(\lambda > 0)$ with some additional assumptions. Recently the results in [**2, 3**] were also improved by D. S. Kang in [**4**] and L. Ding and C. L. Tang in [**5**], respectively. In this paper we discuss (1.1) with a more general $g(x,t)$ by the mountain-pass argument and a linking argument to improve the main results in [**2, 3, 9**]. Roughly $g(x,t)$ has subcritical Sobolev growth.

In view of [**1, 6**] the operator $-\Delta - \dfrac{\mu}{|x|^2}$ $(0 \le \mu < \overline{\mu})$ has discrete spectrum, σ_μ, in $H_0^1(\Omega)$ and each eigenvalue, $\lambda_k (k \ge 1)$, of it is positive, isolated and has finite multiplicity, the smallest eigenvalue λ_1 being simple and $\lambda_k \to +\infty$ as $k \to +\infty$. Furthermore all of its eigenfunctions belong to $H_0^1(\Omega)$.

As in [**1**] for $0 \le \mu < \overline{\mu}$ we endow the Hilbert space, H_μ, with the scalar product

$$(u,v)_{H_\mu} = \int_\Omega \Big(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2}\Big)\,\mathrm{d}x \;\; \forall u, v \in H_\mu$$

and define

$$\|u\|_{H_\mu} = \bigg(\int_\Omega \Big(|\nabla u|^2 - \mu\,\frac{u^2}{|x|^2}\Big)\,\mathrm{d}x\bigg)^{1/2}.$$

We can infer that the norm $\|\cdot\|_{H_\mu}$ is equivalent to the norm in $H_0^1(\Omega)$ by Hardy's inequality.

Define the constant

$$A_{\mu,s}(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \Big(|\nabla u|^2 - \mu\,\frac{u^2}{|x|^2}\Big)\,\mathrm{d}x}{\bigg(\int_\Omega \Big(\frac{|u|^{2^*(s)}}{|x|^s}\Big)\,\mathrm{d}x\bigg)^{2/2^*(s)}}. \tag{1.2}$$

Then $A_{\mu,s}(\Omega)$ is independent of $\Omega \subset \mathbb{R}^N$, see [**7**]. When $s = 0$, $A_{\mu,0}$ is the best Sobolev constant. For simplicity we denote $A_{\mu,s}(\Omega)$ by A in the sequel.

In the paper we need some notation from [**1**]. For fixed $k \in \mathbb{N}$ we denote an L^2 normalized eigenfunction relative to $\lambda_i \in \sigma_\mu$ by e_i, $\forall i \in \mathbb{N}$. We also denote by H^- the space spanned by the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_k$ and $H^+ := (H^-)^\perp$. Take $m \in \mathbb{N}$ such that $B_{1/m} \subset \Omega$ (in the sequel we always assume that), where $B_{1/m} = \{x \in \mathbb{R}^N : |x| < 1/m\}$. Define

$$\zeta_m(x) := \begin{cases} 0 & x \in B_{1/m}, \\ m|x| - 1 & x \in A_m = B_{2/m} \setminus B_{1/m}, \\ 1 & x \in \Omega \setminus B_{2/m}, \end{cases}$$

and $e_i^m := \zeta_m e_i$, $H_m^- := \mathtt{span}\{e_i^m; i = 1, 2, \dots, k\}$.

From [**2**] we know that the functions

$$u_\varepsilon^*(x) = \frac{\mathbb{K}\varepsilon^{\sqrt{\overline{\mu}}/(2-s)}}{|x|^{\sqrt{\overline{\mu}}-\kappa}\left(\varepsilon + |x|^{\frac{2-s}{\sqrt{\overline{\mu}}}\kappa}\right)^{\frac{N-2}{2-s}}}$$

with $\mathbb{K} = \left(\frac{2(\overline{\mu}-\mu)(N-s)}{\sqrt{\overline{\mu}}}\right)^{\frac{\sqrt{\overline{\mu}}}{2-s}}$ and $\kappa = \sqrt{\overline{\mu}-\mu}$, solve the equation $-\Delta u - \mu\,\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}\,u$ in $\mathbb{R}^N \backslash \{0\}$ and $\|u_\varepsilon^*\|^2_{H_\mu(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s}\,\mathrm{d}x = A^{(N-s)/(2-s)}$. Since $u_\varepsilon^*(x)$ is a radial function, we can view it as a function defined on $\mathbb{R}^+$. For all $m \in \mathbb{N}$ and $\varepsilon > 0$ define the shifted functions as [**1, 3**]:

$$u_\varepsilon^m(x) := \begin{cases} u_\varepsilon^*(x) - u_\varepsilon^*(1/m) & x \in B_{1/m}\backslash\{0\}, \\ 0 & x \in \Omega\backslash B_{1/m}. \end{cases} \tag{1.3}$$

In this paper we assume:

(C1) $g(x,t) = \frac{g_1(x,t)}{|x|^s} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $\lim_{t\to\infty} \frac{g(x,t)}{|t|^{2^*-2}t} = 0$ uniformly for *a.e.* $x \in \Omega$;

(C2) $G(x,t) \geq 0$ for *a.e.* $x \in \Omega$ and $\forall t \in \mathbb{R}$, where $G(x,t) = \frac{G_1(x,t)}{|x|^s} = \int_0^t g(x,r)\,\mathrm{d}r = \frac{\int_0^t g_1(x,r)\,\mathrm{d}r}{|x|^s}$;

(C3) there exist positive constants T, a_1, a_2 and ρ satisfying $\frac{a_2}{\beta} + \frac{1}{2^*(s)} \leq \frac{1}{\rho} < \frac{1}{2}$ such that

$$\frac{1}{\rho}\,tg_1(x,t) - G_1(x,t) \geq -a_1|x|^s - a_2|t|^{2^*(s)}, \quad \forall\ a.e.\ x \in \Omega,\ |t| \geq T;$$

(C4) the following hold:

(i) for $0 \leq \mu < \overline{\mu} - 1$ there exist $t_0 > \delta_0 > 0$ and $\eta > 0$ such that

$$G_1(x,t) \geq \eta|x|^s t^2 \quad \text{for } a.e.\ x \in \Omega \quad \text{and} \quad \forall\ |t - t_0| \leq \delta_0;$$

(ii) for $\mu = \overline{\mu} - 1$ there exist $m_0 \in \mathbb{N}, t_0 > \delta_0 > 0, 1 < \ell_0 < \sqrt{\frac{t_0+\delta_0}{t_0}}$ and $\eta > 0$ such that $B_{1/m_0} \subset \Omega$ and

$$G_1(x,t) \geq \eta|x|^s t^2 \quad \text{for } a.e.\ x \in \Omega \text{ and } \forall\ |t - t_0| \leq \delta_0$$

with $\frac{\eta S_N}{4(\sqrt{\overline{\mu}}+\kappa)}\,\frac{1}{\beta^{\frac{N-2}{2-s}}}\mathbb{K}^2\Big(\frac{t_0-\delta_0/2}{t_0}\Big)^2 \ln\Big(\frac{t_0+\delta_0}{\ell_0^2 t_0}\Big) > C_0 m_0^{2^*(s)}$, where $C_0 = S_N\Big(\frac{\mu\mathbb{K}^2}{2} + \frac{\mathbb{K}^{2^*(s)}}{2^*(s)}\Big)\cdot\Big(\frac{1}{2\sqrt{\overline{\mu}-\mu}}+\frac{2}{\sqrt{\overline{\mu}}-\kappa}\Big)\cdot\frac{1}{\beta^{\frac{N-2}{2-s}}}$ and S_N is the surface measure of the unit sphere of $\mathbb{R}^N$;

(iii) for $\overline{\mu}-1 < \mu < \overline{\mu}$ there exist $m_0 \in \mathbb{N}, M > 0$ and $\eta > 0$ such that $B_{1/m_0} \subset \Omega$ and

$$G_1(x,t) \geq \eta |x|^s t^p \quad \text{for } a.e.\ x \in B_{1/m_0} \text{ and } \forall\ |t| \geq M$$

with

$$\eta S_N\Big(1-\frac{1}{p}\Big)^p \frac{\mathbb{K}^p}{(4\beta)^{\frac{(N-2)p}{4-2s}}}\,\frac{1}{Np^{\frac{N}{\sqrt{\overline{\mu}-\kappa}}}} > C_0 m_0^{2^*(s)\sqrt{\overline{\mu}-\mu}},$$

where $p = 2(N-2\sqrt{\overline{\mu}-\mu})/(N-2)$;

(C5) there exist $\alpha \geq 0$ and $\widetilde{C} \geq 0$ such that

$$G_1(x,t) \leq \widetilde{C}|x|^s|t| + \frac{\alpha}{2^*(s)}\,|t|^{2^*(s)} \ \forall\ a.e.\ x \in \Omega \text{ and } \forall\ t \in \mathbb{R};$$

(C5)$'$ there exist $\alpha \geq 0, \theta \in (2, 2^*), \Psi \in L^{q(\theta)}(\Omega)$ and $\nu \geq 0$ such that

$$G_1(x,t) \leq \frac{\nu}{2}\,|x|^s|t|^2 + \Psi(x)|x|^s|t|^\theta + \frac{\alpha}{2^*(s)}\,|t|^{2^*(s)} \ \forall\ a.e.\ x \in \Omega \text{ and } \forall\ t \in \mathbb{R}$$

with $q(\theta) = \frac{2^*}{2^*-\theta}$;

(C6) there exist $\alpha \geq 0, \beta \geq \beta_1 \geq 0, \theta \in (2, 2^*), \Psi \in L^{q(\theta)}(\Omega)$ and $\nu_1 > 0$, $\nu_2 > 0$ such that $\forall\ a.e.\ x \in \Omega$ and $\forall\ t \in \mathbb{R}$

$$\nu_1|x|^s|t| - \beta_1|t|^{2^*(s)-1} \leq |g_1(x,t)| \leq \nu_2|x|^s|t| + \Psi(x)|x|^s|t|^{\theta-1} + \alpha|t|^{2^*(s)-1}$$

with $q(\theta) = \frac{2^*}{2^*-\theta}$. Moreover $tg_1(x,t) \geq 0$.

The following technical condition is also needed:

(H) $\Big(\frac{1}{2a(2^*(s)-1)}\Big)^{(N-2)/(4-2s)} \cdot \frac{2-s}{N+2-2s} > b$, where $a = \frac{\alpha+\beta}{2^*(s)A^{2^*(s)/2}}$ and $b = \Big(\frac{|\Omega|}{\lambda_1}\Big)^{1/2}\cdot\widetilde{C}$, ($\widetilde{C}$ as in (C5)).

It is well known that the nontrivial (weak) solutions of problem (1.1) are equivalent to the nonzero critical points of the functional $J \in C^1(H_\mu, \mathbb{R})$:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x - \frac{\beta}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,\mathrm{d}x.$$

The main results in this paper are:

Theorem 1.1. *Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ *be an open bounded domain with smooth boundary. Assume that for* $0 \leq \mu < \overline{\mu}$

(I) (C1) − (C5) *and* (H)

or

(II) (C1) − (C4) *and* (C5)$'$ *with* $0 \leq \nu < \lambda_1$.

Then (1.1) *admits one positive solution.*

Moreover, if $g(x,t)$ *is odd with* t, *then* (1.1) *has one positive solution and one negative solution.*

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded domain with smooth boundary and assume one of the following three cases holds*:

(I) $N \geq 5$, $0 \leq \mu < \overline{\mu} - 1$ *and* (C1), (C3), (C4) (i), (C6) *with* $\lambda_k < \nu_1 \leq \nu_2 < \lambda_{k+1}$ $(k = 1, 2, \dots)$ *and* $0 \leq \beta_1 \leq \beta$,

(II) $N \geq 8$, $0 \leq \mu < \overline{\mu} - \left(\frac{N+2}{N}\right)^2$ *and* (C1), (C3), (C6) *with* $\lambda_k = \nu_1 \leq \nu_2 < \lambda_{k+1}$ $(k = 1, 2, \dots)$ *and* $\beta_1 = 0$,

(III) $N \geq 8$, $0 \leq \mu < \overline{\mu} - \left(\frac{2N+2-s}{N+2-2^*(s)}\right)^2$ *and* (C1), (C3), (C4) (i), (C6) *with* $\lambda_k = \nu_1 \leq \nu_2 < \lambda_{k+1}$ $(k = 1, 2, \dots)$ *and* $0 < \beta_1 < \beta$.

Then (1.1) *admits one solution which changes sign.*

Moreover, if $g(x,t)$ *is odd with* t, *then* (1.1) *has one pair of sign-changing solutions.*

Remark 1.3. (1). Theorem 1.1 improves the results of [**2, 3**] and Theorem 1.2 improves the results of [**9**].
(2). Here conditions (C4) (i) and (iii) are more general than (2.4) and (2.7) of [**1**].
(3). The condition (C3) is not the same as [**12**].
(4). We of course can assume $\beta = 1$ by the classical "stretching" argument.

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on the mountain-pass argument, see [**1, 8**]. In the sequel we always denote a positive constant by C.

A sequence $\{u_m\} \subset H_\mu$ is said to be a $(PS)_c$ sequence for the functional $J(u)$ if $J(u_m) \to c$ and $J'(u_m) \to 0$ in $(H_\mu)^*$ (the dual space of H_μ).

Lemma 2.1. *Assume* (C1) *and* (C3). *If* $\{u_m\} \subset H_\mu$ *is a* $(PS)_c$ *sequence for* J, *then there exists* $u \in H_\mu$ *such that* $u_m \rightharpoonup u$ *up to a subsequence and* $J'(u) = 0$. *Moreover, if* $c \in \left(0,\ \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}\right)$, *then* $u \neq 0$ *and hence* u *is a nontrivial solution of* (1.1).

Proof. We just sketch the proof for it is similar to that in [**1, 8**]. Since $\{u_m\}$ is a $(PS)_c$ sequence, one can get

$$\begin{aligned} J(u_m) - \frac{1}{\rho}\langle J'(u_m), u_m\rangle = \Big(\frac{1}{2} - \frac{1}{\rho}\Big)\|u_m\|^2_{H_\mu} \\ + \int_\Omega \Big(\frac{1}{\rho}\, g(x,u_m)u_m - G(x,u_m)\Big)\,\mathrm{d}x + \beta\Big(\frac{1}{\rho} - \frac{1}{2^*(s)}\Big)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ = c + o(1). \end{aligned}$$

When one takes (C3) into account, one obtains that $\{u_m\}$ is bounded. Therefore there exists $u \in H_\mu$ such that $u_m \rightharpoonup u$ up to a subsequence and $J'(u) = 0$.

Now we prove the statement $u \neq 0$ if $c \in \left(0,\ \dfrac{2-s}{2(N-s)}\, \dfrac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}\right)$.

We assume that $u \equiv 0$. Then $u_m \rightharpoonup 0$. Since $J'(u_m) \to 0$ in $(H_\mu)^*$, by (C1), one has

$$\|u_m\|^2_{H_\mu} - \beta\int_\Omega \frac{|u_m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x = o(1). \tag{2.1}$$

By (1.2) and using $c > 0$ one obtains $\|u_m\|^2_{H_\mu} \geq \dfrac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} + o(1)$. Then (C1) and (2.1) imply that

$$J(u_m) = \frac{1}{2}\|u_m\|^2_{H_\mu} - \frac{\beta}{2^*(s)}\int_\Omega \frac{|u_m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x + o(1) \geq \frac{2-s}{2(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} + o(1),$$

which contradicts $c < \dfrac{2-s}{2(N-s)}\,\dfrac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}$. □

Lemma 2.2. *Write* $\Gamma = \{P \in C([0,1], H_\mu) | P(0) = 0, J(P(1)) < 0\}$. *If* (C2), (C5) *and* (H) *hold or* (C2) *and* (C5)′ *with* $0 \leq \nu < \lambda_1$ *hold, then* J *admits a* $(PS)_c$ *sequence in the cone of positive functions with* $c = \inf_{P\in\Gamma} \max_{t\in[0,1]} J(P(t))$.

Proof. We prove the statement when (C2), (C5) and (H) hold. The second case is similar. As in Lemma 3 of [**1**] we just need to show that there exist $\sigma > 0$ and $\overline{\rho} > 0$ such that $J(v) \geq \sigma\, \forall v \in \partial B_{\overline{\rho}} \cap H_\mu$.

Indeed (1.2), (C5) and $\|v\|^2_{H_\mu} \geq \lambda_1\|v\|^2_{L^2}$ (see [**1**]) give

$$\begin{aligned} J(v) &= \frac{1}{2}\|v\|^2_{H_\mu} - \int_\Omega G(x,v)\,\mathrm{d}x - \frac{\beta}{2^*(s)}\int_\Omega \frac{|v|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &\geq \frac{1}{2}\|v\|^2_{H_\mu} - \frac{\alpha+\beta}{2^*(s)}\int_\Omega \frac{|v|^{2^*(s)}}{|x|^s}\,\mathrm{d}x - \widetilde{C}\int_\Omega |v|\,\mathrm{d}x \\ &\geq \frac{1}{2}\|v\|^2_{H_\mu} - \frac{\alpha+\beta}{2^*(s)A^{2^*(s)/2}}\|v\|^{2^*(s)}_{H_\mu} - \widetilde{C}\Big(\frac{|\Omega|}{\lambda_1}\Big)^{1/2}\|v\|_{H_\mu} \\ &= \frac{1}{2}\|v\|^2_{H_\mu} - a\|v\|^{2^*(s)}_{H_\mu} - b\|v\|_{H_\mu}. \end{aligned}$$

Hence one can end the proof with (H). □

Lemma 2.3. *For $\varepsilon > 0$ small enough and $m \in \mathbb{N}$ we have*

$$\|u_\varepsilon^m\|^2_{H_\mu} \le A^{(N-s)/(2-s)} + C' \varepsilon^{\frac{N-2}{2-s}} m^{2\sqrt{\bar\mu - \mu}}, \tag{2.2}$$

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \ge A^{(N-s)/(2-s)} - C'' \varepsilon^{\frac{N-2}{2-s}} m^{2^*(s)\sqrt{\bar\mu - \mu}} \tag{2.3}$$

with $C' = \mu S_N \mathbb{K}^2\Big(\frac{1}{2\sqrt{\bar\mu - \mu}} + \frac{2}{\sqrt{\bar\mu} - \kappa}\Big)$ *and* $C'' = S_N \mathbb{K}^{2^*(s)}\Big(\frac{1}{2^*(s)\sqrt{\bar\mu - \mu}} + \frac{2}{\sqrt{\bar\mu} - \kappa}\Big)$.

Proof. The proof is more accurate than the one in Lemma 2.2 of [**9**]. Firstly, if $\mu \neq 0$,

$$\begin{aligned}
&\int_\Omega \frac{(u_\varepsilon^m)^2}{|x|^2}\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \frac{(u_\varepsilon^*)^2}{|x|^2}\,\mathrm{d}x - S_N\mathbb{K}^2 \int_{1/m}^\infty \frac{\varepsilon^{\frac{2\sqrt{\bar\mu}}{2-s}}}{r^{2(\sqrt{\bar\mu}-\kappa)}\Big(\varepsilon + r^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{2(N-2)}{2-s}}} r^{N-3}\,\mathrm{d}r \\
&-2S_N\mathbb{K}^2 \int_0^{1/m} \frac{\varepsilon^{\frac{2\sqrt{\bar\mu}}{2-s}}}{r^{\sqrt{\bar\mu}-\kappa}\Big(\varepsilon + r^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{N-2}{2-s}}\Big(\frac{1}{m}\Big)^{\sqrt{\bar\mu}-\kappa}\Big(\varepsilon + \Big(\frac{1}{m}\Big)^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{N-2}{2-s}}} r^{N-3}\,\mathrm{d}r.
\end{aligned}$$

Since

$$\begin{aligned}
&S_N\mathbb{K}^2 \int_{1/m}^\infty \frac{\varepsilon^{\frac{2\sqrt{\bar\mu}}{2-s}}}{r^{2(\sqrt{\bar\mu}-\kappa)}\Big(\varepsilon + r^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{2(N-2)}{2-s}}} r^{N-3}\,\mathrm{d}r \le \frac{S_N\mathbb{K}^2}{2\sqrt{\bar\mu-\mu}} \varepsilon^{\frac{N-2}{2-s}} m^{2\sqrt{\bar\mu-\mu}}, \\
&2S_N\mathbb{K}^2 \int_0^{1/m} \frac{\varepsilon^{\frac{2\sqrt{\bar\mu}}{2-s}}}{r^{\sqrt{\bar\mu}-\kappa}\Big(\varepsilon + r^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{N-2}{2-s}}\Big(\frac{1}{m}\Big)^{\sqrt{\bar\mu}-\kappa}\Big(\varepsilon + \Big(\frac{1}{m}\Big)^{\frac{2-s}{\sqrt{\bar\mu}}\kappa}\Big)^{\frac{N-2}{2-s}}} r^{N-3}\mathrm{d}r \\
&\qquad\qquad \le \frac{2S_N\mathbb{K}^2}{\sqrt{\bar\mu}-\kappa} \varepsilon^{\frac{N-2}{2-s}} m^{2\sqrt{\bar\mu-\mu}}
\end{aligned}$$

and we have

$$\int_\Omega \frac{(u_\varepsilon^m)^2}{|x|^2}\,\mathrm{d}x \ge \int_{\mathbb{R}^N} \frac{(u_\varepsilon^*)^2}{|x|^2}\,\mathrm{d}x - S_N\mathbb{K}^2\Big(\frac{1}{2\sqrt{\bar\mu-\mu}} + \frac{2}{\sqrt{\bar\mu}-\kappa}\Big)\varepsilon^{\frac{N-2}{2-s}} m^{2\sqrt{\bar\mu-\mu}}$$

.

With $\int_\Omega |\nabla u_\varepsilon^m|^2\,\mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u_\varepsilon^*|^2\,\mathrm{d}x$ (2.2) follows.

Concerning the second inequality one has

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \geq \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s}\,\mathrm{d}x - \int_{\mathbb{R}^N\setminus B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s}\,\mathrm{d}x$$
$$- \int_{B_{1/m}} \frac{2^*(s)|u_\varepsilon^*|^{2^*(s)-1}\mathbb{K}\,\varepsilon^{\frac{\sqrt{\mu}}{2-s}}}{|x|^s\left(\dfrac{1}{m}\right)^{\sqrt{\mu}-\kappa}\left(\varepsilon+\left(\dfrac{1}{m}\right)^{\frac{2-s}{\sqrt{\mu}}\kappa}\right)^{\frac{N-2}{2-s}}}\,\mathrm{d}x$$

with

$$\int_{\mathbb{R}^N\setminus B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s}\,\mathrm{d}x = S_N\mathbb{K}^{2^*(s)}\int_{1/m}^\infty \frac{\varepsilon^{\frac{N-s}{2-s}}}{r^{(\sqrt{\mu}-\kappa)2^*(s)}\left(\varepsilon+r^{\frac{2-s}{\sqrt{\mu}}\kappa}\right)^{\frac{2(N-s)}{2-s}}}\,r^{N-s-1}\,\mathrm{d}r$$
$$\leq \frac{S_N\mathbb{K}^{2^*(s)}}{2^*(s)\sqrt{\mu-\mu}}\,\varepsilon^{\frac{N-s}{2-s}}\,m^{2^*(s)\sqrt{\mu-\mu}}$$

and

$$\int_{B_{1/m}} \frac{2^*(s)|u_\varepsilon^*|^{2^*(s)-1}\mathbb{K}\,\varepsilon^{\frac{\sqrt{\mu}}{2-s}}}{|x|^s\left(\dfrac{1}{m}\right)^{\sqrt{\mu}-\kappa}\left(\varepsilon+\left(\dfrac{1}{m}\right)^{\frac{2-s}{\sqrt{\mu}}\kappa}\right)^{\frac{N-2}{2-s}}}\,\mathrm{d}x$$
$$\leq 2^*(s)S_N\mathbb{K}^{2^*(s)}\varepsilon^{\frac{N-s}{2-s}}\,m^{\sqrt{\mu}+\kappa}\int_0^{1/m} \frac{r^{N-1-s}}{r^{(\sqrt{\mu}-\kappa)(2^*(s)-1)}\left(\varepsilon+r^{\frac{2-s}{\sqrt{\mu}}\kappa}\right)^{\frac{N-2s+2}{2-s}}}\,\mathrm{d}r$$
$$\leq 2^*(s)S_N\mathbb{K}^{2^*(s)}\varepsilon^{\frac{N-s}{2-s}}\,m^{\sqrt{\mu}+\kappa}\int_0^{1/m} \frac{r^{\sqrt{\mu}+(2^*(s)-1)\sqrt{\mu-\mu}-1}}{\dfrac{N-2s+2}{2-s}\,\varepsilon\, r^{2^*(s)\sqrt{\mu-\mu}}}\,\mathrm{d}r$$
$$\leq \frac{2S_N\mathbb{K}^{2^*(s)}}{\sqrt{\mu}-\kappa}\,\varepsilon^{\frac{N-2}{2-s}}\,m^{2\sqrt{\mu-\mu}},$$

where we use the elemental inequality $(a+b)^t \geq tab^{t-1}, a,b>0,\ t\geq 1$. Hence

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x$$
$$\geq \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s}\,\mathrm{d}x - S_N\mathbb{K}^{2^*(s)}\left(\frac{1}{2^*(s)\sqrt{\mu-\mu}}+\frac{2}{\sqrt{\mu}-\kappa}\right)\varepsilon^{\frac{N-2}{2-s}}\,m^{2^*(s)\sqrt{\mu-\mu}}.$$

□

Remark 2.4. If $\sqrt{\mu} > (2^*(s)-1)\kappa$, according to [**9**] inequality (2.3) can be written as

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \geq A^{(N-s)/(2-s)} - C''\varepsilon^{\frac{N-s}{2-s}}\,m^{2^*(s)\sqrt{\mu-\mu}}.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We only prove case (I) since the proof of case (II) is similar to the proof of the first. Firstly we show that problem (1.1) admits one positive solution. By Lemma 2.1 and Lemma 2.2 it is enough to show that there exist $\varepsilon > 0$ small enough and some $m \in \mathbb{N}$ such that

$$\max_{t\geq 0} J(tu_\varepsilon^m) < \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}. \tag{2.4}$$

We proceed by contradiction. Assume that for any $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $t_\varepsilon^m > 0$ such that

$$J(t_\varepsilon^m u_\varepsilon^m) \geq \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}. \tag{2.5}$$

By an argument similar to that in Lemma 5 of [**1**] for any $m \in \mathbb{N}$ we deduce that t_ε^m is bounded as $\varepsilon \to 0$ and $t_\varepsilon^m \to t_0^m > 0$ up to a subsequence.

Claim: $t_0^m = \frac{1}{\beta^{(N-2)/(4-2s)}}$. In fact in the spirit of [**11**] by the contrary, if $t_0^m \neq \frac{1}{\beta^{(N-2)/(4-2s)}}$, the function $f(t) = \frac{1}{2}\,t^2 - \frac{\beta}{2^*(s)}\,t^{2^*(s)}$ $(t > 0)$ reaches its maximum at $t = \frac{1}{\beta^{(N-2)/(4-2s)}}$ and $f\Big(\frac{1}{\beta^{(N-2)/(4-2s)}}\Big) = \frac{2-s}{2(N-s)} \cdot \frac{1}{\beta^{(N-2)/(2-s)}}$. By Lemma 2.3 and (C2) for $\varepsilon > 0$ small enough

$$\begin{aligned}
J(t_\varepsilon^m u_\varepsilon^m) &\leq \frac{1}{2}\,(t_\varepsilon^m)^2 \|u_\varepsilon^m\|_{H_\mu}^2 - \frac{\beta}{2^*(s)} (t_\varepsilon^m)^{2^*(s)} \int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\
&\leq \Big(\frac{1}{2}\,(t_\varepsilon^m)^2 - \frac{\beta}{2^*(s)}\,(t_\varepsilon^m)^{2^*(s)}\Big) A^{(N-s)/(2-s)} \\
&+ \varepsilon^{\frac{N-2}{2-s}} \Big(C' \cdot \frac{1}{2}\,(t_\varepsilon^m)^2 m^{2\sqrt{\bar\mu-\mu}} + C'' \cdot \frac{\beta}{2^*(s)}\,(t_\varepsilon^m)^{2^*(s)} m^{2^*(s)\sqrt{\bar\mu-\mu}}\Big) \\
&< \frac{2-s}{2\,(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}},
\end{aligned}$$

which contradicts (2.5).

The claim above implies

$$\begin{aligned}
&\frac{1}{2}\,(t_\varepsilon^m)^2 \|u_\varepsilon^m\|_{H_\mu}^2 - \frac{\beta}{2^*(s)}\,(t_\varepsilon^m)^{2^*(s)} \int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\
&\leq \frac{2-s}{2\,(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} + \Big(C' \cdot \frac{1}{2}\,(t_\varepsilon^m)^2 + C'' \cdot \frac{\beta}{2^*(s)}\,(t_\varepsilon^m)^{2^*(s)}\Big)\,\varepsilon^{\frac{N-2}{2-s}} m^{2^*(s)\sqrt{\bar\mu-\mu}} \\
&< \frac{2-s}{2\,(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} + C_0\,\varepsilon^{\frac{N-2}{2-s}} m^{2^*(s)\sqrt{\bar\mu-\mu}}
\end{aligned}$$

for $\varepsilon > 0$ small enough.

On the other hand we prove that $\int_\Omega G(x, t_\varepsilon^m u_\varepsilon^m)\,\mathrm{d}x > C_0 \varepsilon^{\frac{N-2}{2-s}} m^{2^*(s)\sqrt{\overline{\mu}-\mu}}$ for $\varepsilon > 0$ small enough and some $m \in \mathbb{N}$.

In order to verify this we distinguish three cases:

(1). $0 \le \mu < \overline{\mu} - 1$. By (C4)(i) to ensure $t_\varepsilon^m u_\varepsilon^m(x) \in [t_0 - \delta_0, t_0 + \delta_0]$ firstly we require that

$$t_\varepsilon^m u_\varepsilon^m(x) \le t_\varepsilon^m u_\varepsilon^*(x) \le \frac{\ell_0}{\beta^{\frac{N-2}{4-2s}}} \cdot \frac{\mathbb{K}\,\varepsilon^{\sqrt{\overline{\mu}}/(2-s)}}{|x|^{\sqrt{\overline{\mu}}+\kappa}} \le t_0 + \delta_0 \ \ \forall x \in B_{1/m} \tag{2.6}$$

for $\varepsilon > 0$ small enough, where $1 < \ell_0 < \sqrt{\dfrac{t_0+\delta_0}{t_0}}$. Hence, if

$$|x| \ge \left(\frac{\ell_0}{\beta^{\frac{N-2}{4-2s}}} \cdot \frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}},$$

then $t_\varepsilon^m u_\varepsilon^m(x) \le t_0 + \delta_0$.

Next for $|x| \ge \left(\dfrac{\ell_0}{\beta^{\frac{N-2}{4-2s}}} \cdot \dfrac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}$ one has

$$\varepsilon |x|^{\frac{(2-s)(\sqrt{\overline{\mu}}-\kappa)}{N-2}} + |x|^{\frac{(2-s)(\sqrt{\overline{\mu}}+\kappa)}{N-2}} \le \left(\frac{t_0}{t_0-\delta_0/2}\right)^{\frac{2-s}{N-2}} |x|^{\frac{(2-s)(\sqrt{\overline{\mu}}+\kappa)}{N-2}}. \tag{2.7}$$

Since $\dfrac{1}{\ell_0 \beta^{\frac{N-2}{4-2s}}} u_\varepsilon^*\Big(\dfrac{1}{m}\Big) + t_0 - \delta_0 < t_0 - \delta_0/2$ for small $\varepsilon > 0$, by (2.7) one can get that, if $\left(\dfrac{\ell_0}{\beta^{\frac{N-2}{4-2s}}} \cdot \dfrac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \le |x| \le \left(\dfrac{1}{\ell_0 \beta^{\frac{N-2}{4-2s}}} \cdot \dfrac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}$, then

$$\begin{aligned} t_\varepsilon^m u_\varepsilon^m(x) = t_\varepsilon^m \Big(u_\varepsilon^*(x) - u_\varepsilon^*\Big(\frac{1}{m}\Big)\Big) &\ge \frac{1}{\ell_0 \beta^{\frac{N-2}{4-2s}}} \left(\frac{\mathbb{K}\varepsilon^{\sqrt{\overline{\mu}}/(2-s)}}{\Big(\frac{t_0}{t_0-\delta_0/2}\Big)|x|^{\sqrt{\overline{\mu}}+\kappa}} - u_\varepsilon^*\Big(\frac{1}{m}\Big)\right) \\ &\ge t_0 - \delta_0/2 - \frac{1}{\ell_0 \beta^{\frac{N-2}{4-2s}}}\, u_\varepsilon^*\Big(\frac{1}{m}\Big) \\ &\ge t_0 - \delta_0. \end{aligned}$$

Having the previous work in hand we have that, if

$$\left(\frac{\ell_0}{\beta^{\frac{N-2}{4-2s}}} \cdot \frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \le |x| \le \left(\frac{1}{\ell_0 \beta^{\frac{N-2}{4-2s}}} \cdot \frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}, \tag{2.8}$$

then $t_\varepsilon^m u_\varepsilon^m(x) \in [t_0 - \delta_0, t_0 + \delta_0]$. Note that from (2.7), (2.8) and $u_\varepsilon^*\Big(\dfrac{1}{m}\Big) \le$

$$\frac{1}{2}u_\varepsilon^*(|x|)\forall|x| \le \left(\frac{1}{\ell_0\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}} \varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \text{ for } \varepsilon>0 \text{ small enough}$$

$$\begin{aligned}\int_\Omega G(x,t_\varepsilon^m u_\varepsilon^m)\,\mathrm{d}x &\ge C\int_{\left(\frac{\ell_0}{\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}^{\left(\frac{1}{\ell_0\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}} (u_\varepsilon^*(r))^2 r^{N-1}\,\mathrm{d}r\\
&\ge C\varepsilon^{\frac{N-2}{2-s}}\int_{\left(\frac{\ell_0}{\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}^{\left(\frac{1}{\ell_0\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}} r^{1-2\sqrt{\overline{\mu}-\mu}}\,\mathrm{d}r\\
&= C\varepsilon^{\frac{N-2}{2-s}}\varepsilon^{\frac{\sqrt{\overline{\mu}}(2-2\sqrt{\overline{\mu}-\mu})}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}\\
&> C_0\varepsilon^{\frac{N-2}{2-s}} m^{2^*(s)\sqrt{\overline{\mu}-\mu}}\end{aligned}$$

for $\varepsilon>0$ small enough.

(2). $\mu=\overline{\mu}-1$. Case (1) shows that, if

$$\left(\frac{\ell_0}{\beta^{\frac{N-2}{4-2s}}}\cdot\frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \le |x| \le \left(\frac{1}{\ell_0\beta^{\frac{N-2}{4-2s}}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \tag{2.9}$$

for $\varepsilon>0$ small enough, then $t_\varepsilon^{m_0}u_\varepsilon^{m_0}(x)\in[t_0-\delta_0,t_0+\delta_0]$. Therefore, noting (2.7) and $u_\varepsilon^*\left(\frac{1}{m_0}\right)\le\frac{1}{2}u_\varepsilon^*(|x|)\forall|x|\le\left(\frac{1}{\ell_0\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}$ for $\varepsilon>0$ small enough, by (C4)(ii), one has

$$\begin{aligned}&\int_\Omega G(x,t_\varepsilon^{m_0}u_\varepsilon^{m_0})\,\mathrm{d}x\\
&\ge\frac{\eta S_N}{4}\left(\frac{1}{\beta^{\frac{N-2}{2-s}}}+o(1)\right)\int_{\left(\frac{\ell_0}{\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}^{\left(\frac{1}{\ell_0\beta^{(N-2)/(4-2s)}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}(u_\varepsilon^*(r))^2r^{N-1}\,\mathrm{d}r\\
&\ge\frac{\eta S_N}{4}\left(\frac{1}{\beta^{\frac{N-2}{2-s}}}+o(1)\right)\mathbb{K}^2\left(\frac{t_0-\delta_0/2}{t_0}\right)^2\varepsilon^{\frac{N-2}{2-s}}\int_{\left(\frac{\ell_0}{\beta^{\frac{N-2}{4-2s}}}\cdot\frac{\mathbb{K}}{t_0+\delta_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}^{\left(\frac{1}{\ell_0\beta^{\frac{N-2}{4-2s}}}\cdot\frac{\mathbb{K}}{t_0}\right)^{\frac{1}{\sqrt{\overline{\mu}}+\kappa}}\varepsilon^{\frac{\sqrt{\overline{\mu}}}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}}}r^{-1}\mathrm{d}r\\
&=\frac{\eta S_N}{4(\sqrt{\overline{\mu}}+\kappa)}\left(\frac{1}{\beta^{(N-2)(2-s)}}+o(1)\right)\mathbb{K}^2\left(\frac{t_0-\delta_0/2}{t_0}\right)^2\ln\left(\frac{t_0+\delta_0}{\ell_0^2t_0}\right)\varepsilon^{(N-2)/(2-s)}\\
&>C_0\varepsilon^{(N-2)/(2-s)}m_0^{2^*(s)\sqrt{\overline{\mu}-\mu}}\end{aligned}$$

for $\varepsilon>0$ small enough.

(3). $\overline{\mu}-1<\mu<\overline{\mu}$. The proof is similar to the former part of Lemma 6 in [**1**] and we simply sketch it here. Let $\kappa' = \dfrac{\sqrt{\overline{\mu}}}{\kappa(2-s)}$. Then

$$\varepsilon |x|^{\frac{(2-s)(\sqrt{\overline{\mu}}-\kappa)}{N-2}} + |x|^{\frac{(2-s)(\sqrt{\overline{\mu}}+\kappa)}{N-2}} \le 2\,\varepsilon^{\frac{(2-s)(\sqrt{\overline{\mu}}+\kappa)}{N-2}\kappa'}\,\forall x\in B_{\varepsilon^{\kappa'}}\subset B_{1/m_0} \tag{2.10}$$

for $\varepsilon>0$ small enough. On the other hand we have

$$t_\varepsilon^{m_0}u_\varepsilon^{m_0}(x)\ge M\,\forall x\in B_{\varepsilon^{\kappa'}} \tag{2.11}$$

and

$$u_\varepsilon^*(x)\ge p u_\varepsilon^*(1/m_0)\,\forall x\in B_{\varepsilon^{\kappa'}/q'} \tag{2.12}$$

with $q'=p^{1/(\sqrt{\overline{\mu}}-\kappa)}$ for $\varepsilon>0$ small enough.

Combining (2.10), (2.11), (2.12) and (C4)(ii) we obtain

$$\begin{aligned}
\int_\Omega G(x,t_\varepsilon^{m_0}u_\varepsilon^{m_0})\,\mathrm{d}x &\ge \eta S_N\Big(1-\frac{1}{p}\Big)^p\Big(\frac{1}{\beta^{\frac{(N-2)p}{4-2s}}}+o(1)\Big)\int_0^{\varepsilon^{\kappa'}/q'}(u_\varepsilon^*(r))^p r^{N-1}\,\mathrm{d}r\\
&\ge \eta S_N\Big(1-\frac{1}{p}\Big)^p\Big(\frac{1}{\beta^{\frac{(N-2)p}{4-2s}}}+o(1)\Big)\frac{\mathbb{K}^p}{2^{\frac{(N-2)p}{2-s}}}\,\varepsilon^{\frac{-p\overline{\mu}}{\kappa(2-s)}}\int_0^{\varepsilon^{\kappa'}/q'} r^{N-1}\,\mathrm{d}r\\
&= \eta S_N\Big(1-\frac{1}{p}\Big)^p\Big(\frac{1}{\beta^{\frac{(N-2)p}{4-2s}}}+o(1)\Big)\frac{\mathbb{K}^p}{2^{\frac{(N-2)p}{2-s}}}\,\frac{1}{Np^{\frac{N}{\sqrt{\overline{\mu}}-\kappa}}}\,\varepsilon^{\frac{N-2}{2-s}}\\
&> C_0\,\varepsilon^{\frac{N-2}{2-s}}\,m_0^{2^*(s)\sqrt{\overline{\mu}-\mu}}
\end{aligned}$$

for $\varepsilon>0$ small enough.

We now conclude that problem (1.1) admits one positive solution since

$$\begin{aligned}
J(t_\varepsilon^m u_\varepsilon^m) = {}& \frac{1}{2}\,\|t_\varepsilon^m u_\varepsilon^m\|_{H_\mu}^2-\int_\Omega G(x,t_\varepsilon^m u_\varepsilon^m)\,\mathrm{d}x-\frac{\beta}{2^*(s)}\int_\Omega\frac{|t_\varepsilon^m u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x\\
&< \frac{2-s}{2(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}
\end{aligned}$$

for $\varepsilon>0$ small enough($m=m_0$ in case (2) and case (3)), which contradicts (2.5).

Moreover, if $g(x,t)$ is odd with t, then $-u$ is one negative solution of (1.1). □

3. PROOF OF THEOREM 1.2

We begin this Section with two lemmas.

Lemma 3.1 *Assume* (C6) *with* $\nu_1>\lambda_1$ *or* $\nu_1=\lambda_1$ *and* $0\le\beta_1<\beta$. *Then every nontrivial solution of* (1.1) *must be sign-changing.*

Proof. By the contrary we assume that $u \geq 0$ is a nontrivial solution of (1.1). We have

$$-\int_\Omega \Delta u e_1 - \mu \int_\Omega \frac{u}{|x|^2}\, e_1 = \int_\Omega g(x,u) e_1 + \beta \int_\Omega \frac{|u|^{2^*(s)-2}}{|x|^s}\, u e_1$$

and

$$-\int_\Omega \Delta u e_1 - \mu \int_\Omega \frac{u}{|x|^2}\, e_1 = \int_\Omega u\Big(- \Delta e_1 - \frac{\mu}{|x|^2}\, e_1\Big) = \lambda_1 \int_\Omega u e_1.$$

(C6) and the above two equations imply that

$$\lambda_1 \int_\Omega u e_1 \geq \nu_1 \int_\Omega u e_1 + (\beta - \beta_1) \int_\Omega \frac{|u|^{2^*(s)-2}}{|x|^s}\, u e_1.$$

Therefore, if $\nu_1 > \lambda_1$ or $\nu_1 = \lambda_1$ and $0 \leq \beta_1 < \beta$, we can get a contradiction. Then (1.1) has no nontrivial positive solutions. Similar arguments show that (1.1) has no nontrivial negative solutions. □

By (C6) we find that for $a.e. x \in \Omega$ and $\forall t \in \mathbb{R}$

$$\frac{\nu_1}{2}\, |x|^s |t|^2 - \frac{\beta_1}{2^*(s)}\, |t|^{2^*(s)} \leq |G_1(x,t)| \leq \frac{\nu_2}{2}\, |x|^s |t|^2 + \frac{\Psi(x)}{\theta}\, |x|^s |t|^\theta + \frac{\alpha}{2^*(s)}\, |t|^{2^*(s)}. \tag{3.1}$$

Lemma 3.2. *Assume* (C6) *with* $\lambda_k < \nu_1 \leq \nu_2 < \lambda_{k+1}$ *or* $\lambda_k = \nu_1 \leq \nu_2 < \lambda_{k+1}$ *and* $0 \leq \beta_1 < \beta$. *Let* $Q_m^\varepsilon := [(\overline{B_R} \cap H_m^-) \bigoplus [0,\ R]\{u_\varepsilon^m\}]$ *and* $\Gamma := \{h \in C(Q_m^\varepsilon, H_\mu) : h(v) = v, \forall v \in \partial Q_m^\varepsilon\}$. *Then* J *admits a* $(PS)_c$ *sequence at level*

$$c = \inf_{h\in\Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)).$$

Proof. See the proof of Lemma 4 in [**1**] (see also [**3, 4**]). □

The proof of Theorem 1.2 is the following.

Proof of Theorem 1.2. Since the identity $Id \in \Gamma$, we have

$$\inf_{h\in\Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)) \leq \max_{v \in Q_m^\varepsilon} J(v).$$

Theorem 1.2 follows from Lemma 2.1, Lemma 3.1 and Lemma 3.2 if one can prove that for some $\varepsilon > 0$ and $m \in \mathbb{N}$

$$\sup_{v \in Q_m^\varepsilon} J(v) < \frac{2-s}{2\,(N-s)}\, \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}\,. \tag{3.2}$$

On the contrary we assume that

$$\forall \varepsilon > 0 \,\text{and}\, \forall m \in \mathbb{N} \quad \sup_{v \in Q_m^\varepsilon} J(v) \geq \frac{2-s}{2(N-s)}\, \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}\,. \tag{3.3}$$

One notes that $\{v \in Q^\varepsilon_m; J(v) \ge 0\}$ is compact. The supremum in (3.3) is attained. Thus for all $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists $w^m_\varepsilon \in H^-_m$ and $t^m_\varepsilon \ge 0$ such that for $v^m_\varepsilon = w^m_\varepsilon + t^m_\varepsilon u^m_\varepsilon$ we have

$$J(v^m_\varepsilon) = \max_{v \in Q^\varepsilon_m} J(v) \ge \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}. \tag{3.4}$$

Similarly to [**1,4**], for any $m \in \mathbb{N}$, $\{t^m_\varepsilon\} \subset \mathbb{R}^+$ and $\{w^m_\varepsilon\} \subset H^-_m$ are bounded. Up to subsequences we assume that $t^m_\varepsilon \to t^m \ge 0$, $\; w^m_\varepsilon \to w^m \in H^-_m$.

To obtain a contradiction to (3.4) we distinguish three cases according to the assumptions of Theorem 1.2.

Case (I). Using $\max\limits_{\{u \in H^-_m; \|u\|_{L^2(\Omega)}=1\}} \|u\|^2_{H_\mu} \le \lambda_k + Cm^{-2\sqrt{\bar\mu-\mu}}$ (for details see [**10**]) and (3.1) we know that

$$\begin{aligned} J(w^m_\varepsilon) &= \frac{1}{2}\,\|w^m_\varepsilon\|^2_{H_\mu} - \int_\Omega G(x, w^m_\varepsilon)\,\mathrm{d}x - \frac{\beta}{2^*(s)} \int_\Omega \frac{|w^m_\varepsilon|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &\le \frac{\lambda_k + Cm^{-2\sqrt{\bar\mu-\mu}}}{2}\,\|w^m_\varepsilon\|^2_{L^2} - \frac{\nu_1}{2}\,\|w^m_\varepsilon\|^2_{L^2} - \frac{\beta-\beta_1}{2^*(s)} \int_\Omega \frac{|w^m_\varepsilon|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \le 0 \end{aligned}$$

for m large enough. On the other hand, as we see in the proof of Theorem 1.1, we have

$$J(t^m_\varepsilon u^m_\varepsilon) < \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}$$

for $\varepsilon > 0$ small enough and m large enough. Then the above two inequalities with $J(v^m_\varepsilon) = J(w^m_\varepsilon) + J(t^m_\varepsilon u^m_\varepsilon)$ imply a contradiction to (3.4).

Case (II). By $\max\limits_{\{u \in H^-_m; \|u\|_{L^2(\Omega)}=1\}} \|u\|^2_{H_\mu} \le \lambda_k + Cm^{-2\sqrt{\bar\mu-\mu}}$ and (3.1), noting H^-_m is finite dimensional and then the convergence of w^m_ε can be viewed as in any norm topology, we see that

$$\begin{aligned} J(w^m_\varepsilon) &= \frac{1}{2}\,\|w^m_\varepsilon\|^2_{H_\mu} - \int_\Omega G(x, w^m_\varepsilon)\,\mathrm{d}x - \frac{\beta}{2^*(s)} \int_\Omega \frac{|w^m_\varepsilon|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &\le \frac{Cm^{-2\sqrt{\bar\mu-\mu}}}{2}\,\|w^m_\varepsilon\|^2_{L^2} - \frac{\beta}{2^*(s)} \int_\Omega \frac{|w^m_\varepsilon|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &= C_1 m^{-2\sqrt{\bar\mu-\mu}}\|w^m_\varepsilon\|^2_{L^2} - C_2\|w^m_\varepsilon\|^{2^*(s)}_{L^2} \\ &\le C_3 m^{-\frac{2(N-s)}{2-s}\sqrt{\bar\mu-\mu}}. \end{aligned}$$

As was done in [**3, 9**], setting $\varepsilon = m^{-\frac{(N+2)(2-s)\kappa}{N-2}}$ we denote $v^m_\varepsilon, t^m_\varepsilon, u^m_\varepsilon, w^m_\varepsilon$ by v^m, t^m, u^m, w^m, respectively, in the sequel. Now we estimate $J(t^m u^m)$. Clearly t^m is bounded and $t^m \to t^0 > 0$ up to a subsequence. Moreover (2.2) and (2.3) become

$$\|u^m_\varepsilon\|^2_{H_\mu} \le A^{(N-s)/(2-s)} + C' m^{-N\sqrt{\bar\mu-\mu}}, \tag{3.5}$$

(3.6) $$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \ge A^{(N-s)/(2-s)} - C''m^{-((N+2)-2^*(s))\sqrt{\overline{\mu}-\mu}}.$$

When $\sqrt{\overline{\mu}} > (2^*(s)-1)\kappa$, that is $\mu > \overline{\mu} - \dfrac{(N-2)^4}{4(N+2-2s)^2}$, according to [**9**] (3.6) can be replaced by

(3.7) $$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \ge A^{(N-s)/(2-s)} - C''m^{\frac{-N(N-s)}{N-2}\sqrt{\overline{\mu}-\mu}}.$$

By (3.1) one has

(3.8) $$J(t^m u^m) \le \frac{1}{2}\,\|t^m u^m\|^2_{H_\mu} - \frac{\nu_1}{2}\int_\Omega |t^m u^m|^2\,\mathrm{d}x - \frac{\beta}{2^*(s)}\int_\Omega \frac{|t^m u^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x.$$

With $\nu_1\int_\Omega |u^m|^2\,\mathrm{d}x \ge C_4 m^{-(N+2)}$ (for details see [**1, 9**]) we know:

(i). For $\overline{\mu} - \dfrac{(N-2)^4}{4(N+2-2s)^2} < \mu < \overline{\mu} - \left(\dfrac{N+2}{N}\right)^2$

$$\begin{aligned} J(t^m u^m) &\le \frac{1}{2}\,\|t^m u^m\|^2_{H_\mu} - \frac{\nu_1}{2}\int_\Omega |t^m u^m|^2\,\mathrm{d}x - \frac{\beta}{2^*(s)}\int_\Omega \frac{|t^m u^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &\le \frac{1}{2}\,(t^m)^2 (A^{(N-s)/(2-s)} + C'm^{-N\sqrt{\overline{\mu}-\mu}} - C_4 m^{-(N+2)}) \\ &\qquad - \frac{\beta}{2^*(s)}\,(t^m)^{2^*(s)}\Big(A^{(N-s)/(2-s)} - C''m^{\frac{-N(N-s)}{N-2}\sqrt{\overline{\mu}-\mu}}\Big) \\ &\le \frac{2-s}{2(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_5 m^{-(N+2)}. \end{aligned}$$

With $J(v^m) = J(w^m) + J(t^m u^m)$ (by the fact $|\mathtt{supp}(u^m)\cap\mathtt{supp}(w^m)| = 0$) we get

$$\begin{aligned} J(v^m) &\le \frac{2-s}{2(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_5 m^{-(N+2)} + C_3 m^{-\frac{2(N-s)}{2-s}\sqrt{\overline{\mu}-\mu}} \\ &< \frac{2-s}{2(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} \end{aligned}$$

for m large enough, which implies a contradiction to (3.4).

(ii). For $0 \le \mu \le \overline{\mu} - \dfrac{(N-2)^4}{4(N+2-2s)^2}$

$$\begin{aligned} J(t^m u^m) &\le \frac{1}{2}\,\|t^m u^m\|^2_{H_\mu} - \frac{\nu_1}{2}\int_\Omega |t^m u^m|^2\,\mathrm{d}x - \frac{\beta}{2^*(s)}\int_\Omega \frac{|t^m u^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x \\ &\le \frac{1}{2}\,(t^m)^2 (A^{(N-s)/(2-s)} + C'm^{-N\sqrt{\overline{\mu}-\mu}} - C_4 m^{-(N+2)}) \\ &- \frac{\beta}{2^*(s)}\,(t^m)^{2^*(s)}(A^{(N-s)/(2-s)} - C''m^{-(N+2-2^*(s))\sqrt{\overline{\mu}-\mu}}) \\ &\le \frac{2-s}{2\,(N-s)}\,\frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_6 m^{-(N+2)}. \end{aligned}$$

Then, as we did for (i), one obtains

$$J(v^m) \le \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_6 m^{-(N+2)} + C_3 m^{-\frac{2(N-s)}{2-s}\sqrt{\overline{\mu}-\mu}}$$
$$< \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}}$$

for m large enough, which contradict (3.4).

Case (III). When we use (C4)(i), the proof of case (1) of Theorem 1.1 gives

$$\int_\Omega G(x, t_\varepsilon^m u_\varepsilon^m)\,\mathrm{d}x \ge C_7\,\varepsilon^{\frac{N-2}{2-s}}\,\varepsilon^{\frac{\sqrt{\overline{\mu}}(2-2\sqrt{\overline{\mu}-\mu})}{(\sqrt{\overline{\mu}}+\kappa)(2-s)}} \tag{3.9}$$

for $\varepsilon > 0$ small enough.

Setting $\varepsilon = m^{-\frac{(N+2)(2-s)\kappa}{N-2}}$ as in *Case* (II), since $0 \le \mu < \overline{\mu} - \Big(\dfrac{2N+2-s}{N+2-2^*(s)}\Big)^2$, we have

$$J(t^m u^m) \le \frac{1}{2}\,\|t^m u^m\|^2_{H_\mu} - \int_\Omega G(x, t^m u^m)\,\mathrm{d}x - \frac{\beta}{2^*(s)} \int_\Omega \frac{|t^m u^m|^{2^*(s)}}{|x|^s}\,\mathrm{d}x$$
$$\le \frac{1}{2}\,(t^m)^2\,\big(A^{(N-s)/(2-s)} + C' m^{-N\sqrt{\overline{\mu}-\mu}}\big) - C_7 m^{-\frac{N(N+2)\kappa}{N-2+2\kappa}}$$
$$- \frac{\beta}{2^*(s)}\,(t^m)^{2^*(s)}\big(A^{(N-s)/(2-s)} - C'' m^{-(N+2-2^*(s))\sqrt{\overline{\mu}-\mu}}\big)$$
$$\le \frac{2-s}{2\,(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_8 m^{-\frac{N(N+2)\kappa}{N-2+2\kappa}}.$$

On the other hand *Case* (II) shows that

$$J(w^m) \le C_9 m^{-\frac{2(N-s)}{2-s}\kappa}. \tag{3.10}$$

The above two inequalities with $J(v^m) = J(w^m) + J(t^m u^m)$ imply

$$J(v^m) \le \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}} - C_8 m^{-\frac{N(N+2)\kappa}{N-2+2\kappa}} + C_9 m^{-\frac{2(N-s)}{2-s}\kappa}$$
$$< \frac{2-s}{2(N-s)} \frac{A^{(N-s)/(2-s)}}{\beta^{(N-2)/(2-s)}},$$

which implies a contradiction to (3.4) for m large enough.

In conclusion (1.1) admits one sign-changing solution u. Moreover, if $g(x,t)$ is odd in t, then $-u$ is also a sign-changing solution of (1.1). □

Remark 3.3. (1). From the proof of (II)(ii) we know that the theorem still holds for $N = 5, 6, 7$ if $\overline{\mu} - \Big(\dfrac{N+2}{N+2-2^*(s)}\Big)^2 > 0$ for some $0 \le s < 2$ and $\mu \in \Big[0, \overline{\mu} - \Big(\dfrac{N+2}{N+2-2^*(s)}\Big)^2\Big)$.

(2). From the proof of (III) we see that the theorem also holds for $N = 5, 6, 7$ if $\overline{\mu} - \Big(\frac{2N+2-s}{N+2-2^*(s)}\Big)^2 > 0$ for some $0 \le s < 2$ and $\mu \in \Big[0, \overline{\mu} - \Big(\frac{2N+2-s}{N+2-2^*(s)}\Big)^2\Big)$.

Acknowledgements. The authors thank the anonymous referee for very helpful suggestions and comments.

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(Received November 12, 2007)
(Revised February 8, 2008)