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LEVEL GENERATING TREES AND PROPER RIORDAN ARRAYS

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We introduce a generalization of generating trees named Level Generating Trees and study the connection between these structures and proper Riordan arrays, deriving a theorem that, under suitable conditions, associates a Riordan array to a Level Generating Tree and vice versa. We illustrate our main results by several examples concerning classical combinatorial structures.

1. INTRODUCTION

The concept of generating trees has been introduced in the literature by Chung, Graham, Hoggat and Kleiman in [**4**] to examine Baxter permutations. This technique has been successfully applied by West [**17**, **18**] to other classes of permutations and more recently to some other combinatorial classes such as plane trees and lattice paths (see Barcucci et al. [**2**]). In all these cases, a generating tree is associated to a certain combinatorial class, according to some enumerative parameter, in such a way that the number of nodes appearing on level n of the tree gives the number of n-sized objects in the class.

If a problem has been defined by means of a generating tree, some device has to be used to obtain counting information on the objects of the associated combinatorial class. In [**11**] and [**9**], Merlini, Sprugnoli and Verri have introduced the concept of matrix associated to a generating tree (AGT matrix, for short): this is an infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ where $d_{n,k}$ is the number of nodes at level n with label $k+c$, c being the root label. The main result in [**9**] is Theorem 3.3 (3.1 in this paper) which states the conditions under which an AGT matrix is a proper Riordan array, and vice versa. For example, the following labeled tree (this example concerns only non-marked nodes)

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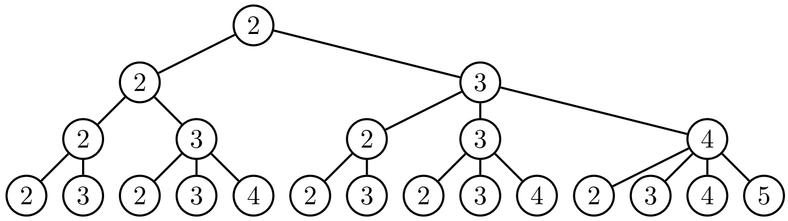


corresponds to the generating tree, up to level 3, defined by the specification:

$$\begin{cases} root : (2) \\ rule : (k) \to (2) \dots (k)(k+1). \end{cases} \tag{1.1}$$

This is known as the *Catalan* generating tree: in fact, the corresponding AGT matrix, shown in Table 1, is strictly related to the generating function of CATALAN numbers, $C(t) = (1 - \sqrt{1-4t})/(2t)$. The generic element $C_{n,k}$ in the array is given by:

$$C_{n,k} = [t^{n-k}]C(t)^{k+1},$$

that is, $(C_{n,k})$ is the RIORDAN array $(C(t), tC(t))$.

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	5	5	3	1			
4	14	14	9	4	1		
5	42	42	28	14	5	1	
6	132	132	90	48	20	6	1

TABLE 1. The Catalan triangle.

The concept of a RIORDAN array provides a remarkable characterization of many lower triangular arrays that arise in combinatorics and algorithm analysis. The theory of *Riordan arrays* has been introduced in 1991 by SHAPIRO, GETU, WOAN and WOODSON [**15**], with the aim of generalizing the concept of a *renewal array* defined by ROGERS [**12**] in 1978. Their basic idea was to define a group of infinite lower triangular arrays with properties analogous to those of the PASCAL triangle. This concept has also been studied by SPRUGNOLI [**16**], who pointed out the relevance of these matrices from a theoretical and practical point of view. A *Riordan array* is a pair $(d(t), h(t))$ in which $d(t)$ and $h(t)$ are formal power series; if $d(0) \neq 0$ and $h(0) = 0, h'(0) \neq 0$, the RIORDAN array is called *proper*. The pair defines an infinite, lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ where $d_{n,k} = [t^n]d(t)h(t)^k$.

In the recent literature, RIORDAN arrays have attracted the attention of various authors and many examples and applications can be found [**1**, **6**, **8**, **10**, **11**, **14**, **19**, **20**, **21**]. Most of them deal with the original formulation of RIORDAN arrays, that is, in the corresponding matrices each element $d_{n+1,k+1}$ is given by a linear combination of the elements in the previous row, starting from the previous column (see Theorem 2.1). The coefficients of this linear combination are independent of n and k, and therefore they constitute a specific sequence $(a_0, a_1, a_2, \ldots)$, called the A-sequence of the RIORDAN array. For example, in the CATALAN triangle we have $A = (1, 1, 1, \ldots)$, in fact $C_{n+1,k+1} = \sum_{j \geq k} C_{n,j}$, as can be easily verified from Table 1. However, several new characterizations of RIORDAN arrays have been given in [**7**]: the main result in that paper shows that a lower triangular array $(d_{n,k})$ is RIORDAN whenever its generic element $d_{n+1,k+1}$ linearly depends on the elements $(d_{r,s})$ lying in a well-defined, but large zone of the array; the corresponding coefficients constitute the so-called A-matrix (see Theorem 2.3 and Figure 2 in the present paper). There is no difference between RIORDAN arrays defined in either way: the A-sequence is a particular case of A-matrix and, given a RIORDAN array defined by an A-matrix, this corresponds to a well defined A-sequences. This fact provides a remarkable characterization of many lower triangular arrays for which a recurrence can be given involving elements belonging to the relevant zone.

There are some examples in which a RIORDAN array can be easily studied by means of the A-matrix while the A-sequence is very complex. From a combinatorial point of view, this means that it is very challenging to find a construction allowing to obtain objects of size $n+1$ from objects of size n. Instead, the existence of a simple A-matrix corresponds to a possible construction from objects of different sizes less than $n+1$. For example, in Section 4 a combinatorial problem is described by an A-matrix, containing *simple coefficients* (i.e. integer values). In fact, we take into consideration the RIORDAN array defined by:

$$R(t, w) = \frac{1-t}{1 - t(1+w) - t^2(1-w)},$$

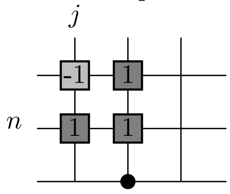
which corresponds to a rather complicated A-sequence:

$$\begin{aligned} A(t) &= \frac{1 + t + \sqrt{(1+t)^2 + 4t(t-1)}}{2} \\ &= 1 + t^2 + t^3 - 2t^5 - 3t^6 + t^7 + 11t^8 + 15t^9 + O(t^{10}). \end{aligned}$$

This corresponds to the recurrence:

$$d_{n+1,k+1} = d_{n,k} + d_{n,k+2} + d_{n,k+3} - 2d_{n,k+5} - 3d_{n,k+6} + d_{n,k+7} + 11d_{n,k+8} + \cdots,$$

whereas, the associated A-matrix can be represented as follows:



and corresponds to the simple recurrence:

$$d_{n+1,k+1} = d_{n,k} + d_{n,k+1} - d_{n-1,k} + d_{n-1,k+1}.$$

In order to treat these cases, we propose a new type of generating tree corresponding to the concept of A-matrix. In this structure, called Level Generating Tree (LGT for short), a node can generate children, grandchildren, great grandchildren, ... nodes (next levels) and brother nodes (same level). Our main result is a theorem which states the conditions under which a matrix associated to a LGT is a proper RIORDAN array, and vice versa. The aim of this paper is to introduce the concept

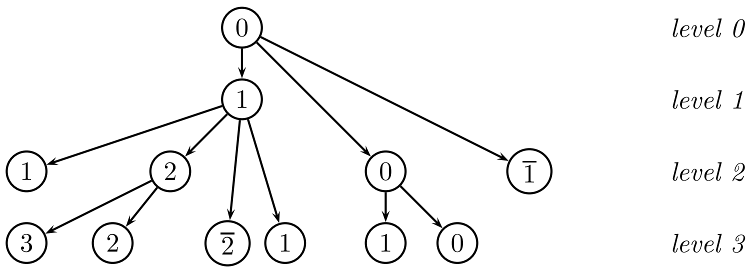


FIGURE 1. A LGT example.

of level generating trees corresponding to RIORDAN arrays whose generic element $d_{n,k}$ depends in a "simple way" from the elements of several previous rows. This generalizes the usual concept of generating trees corresponding to dependencies from the previous row only. We wish to describe some examples, in which the problem can be easily studied by means of the A-matrix. In these examples the A-sequence is very complex, so that approaching the problem with just this sequence seems almost impossible.

The structure of the paper is as follows. In Section 2 we summarize the main results concerning RIORDAN arrays. In Section 3 we describe the concept of Level Generating Trees and prove Theorem 3.2. In the remaining Sections, we study some significant examples with the new LGT concept: the BLOOM's strings, some lattice path problems and binary words excluding a pattern.

2. RIORDAN ARRAYS

A *Riordan array* is a pair $(d(t), h(t))$ in which $d(t)$ and $h(t)$ are formal power series; if $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$, the RIORDAN array is called *proper*. The pair defines an infinite, lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ where:

$$d_{n,k} = [t^n]d(t)h(t)^k.$$

From this definition, it easily follows that $d(t)h(t)^k$ is the generating function of column k in the array. Therefore,

$$D(t,w) = \frac{d(t)}{1 - wh(t)}$$

is the bivariate generating function of the triangle. The RIORDAN array theory allows us to find properties concerning these matrices; for example, we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(h(t)), \tag{2.1}$$

for every sequence f_k having $f(t)$ as its generating function. A description of the RIORDAN array theory, together with many examples, can be found in SHAPIRO et al. [**15**] or in SPRUGNOLI [**16**]. ROGERS [**12**] observed the following, fundamental characterization of proper RIORDAN arrays:

Theorem 2.1. *An array $(d_{n,k})_{n,k\in\mathbb{N}}$ is a proper Riordan array if and only if there exists a sequence $A = (a_i)_{i\in\mathbb{N}}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ (not lying in column 0 or row 0) can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column, i.e.:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots . \tag{2.2}$$

The sum in (2.2) is actually finite because $d_{n,k} = 0$, $\forall k > n$. Sequence A, called the A-*sequence* of the RIORDAN array, is characteristic in the sense that it determines (and is determined by) the function $h(t)$. If $A(t)$ is the generating function of the A-sequence, it can be proven (see SPRUGNOLI [**16**]) that $h(t)$ is the solution of the functional equation:

$$h(t) = tA(h(t)). \tag{2.3}$$

The A-sequence does not completely characterize a proper RIORDAN array $(d(t), h(t))$ because the function $d(t)$ is independent of $A(t)$. In [**7**] the following new characterizations have been proved:

Theorem 2.2. *Let $(d_{n,k})_{n,k\in\mathbb{N}}$ be any infinite lower triangular array with $d_{n,n} \neq 0, \forall n \in \mathbb{N}$ (in particular, let it be a proper Riordan array $(d(t), h(t))$; then a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ exists (called the Z-sequence of the array) such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.:*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots .$$

Let $Z(t)$ be the generating function of the Z-sequence; then:

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}. \tag{2.4}$$

The relation can be inverted and this gives us a formula for the Z-sequence:

$$Z(y) = \left[\frac{d(t) - d_{0,0}}{td(t)} \;\middle|\; y = h(t) \right]. \tag{2.5}$$

The following theorems show that we can characterize a RIORDAN array by means of an A-matrix, rather than by a simple A-sequence. For a possible generalization the reader can refer to the paper [**7**].

Theorem 2.3. *A lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ is Riordan if and only if there exist another array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and a sequence $(\rho_j)_{j\in\mathbb{N}}$ such that*:

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d_{n-i,k+j} + \sum_{j\geq 0}\rho_j d_{n+1,k+j+2}. \tag{2.6}$$

In Figure 2, we try to give a graphic representation of the zones from which the generic element $d_{n+1,k+1}$ (denoted by a small disk or bullet) is allowed to depend, so that the array is Riordan. The only restrictions are that $\alpha_{0,0} \neq 0$. In [**7**] a further characterization has been proved.

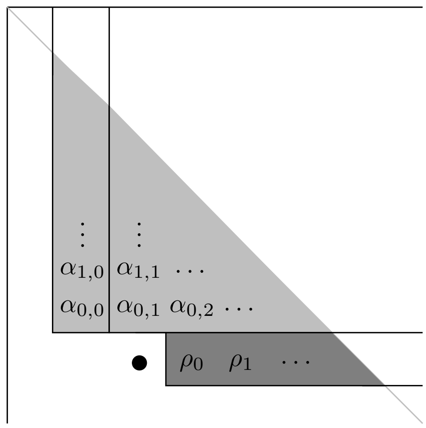


Figure 2. The zones which $d_{n+1,k+1}$ can depend on.

As previously noted, the A-sequence and the function $h(t)$ of a Riordan array are strictly related to each other. This fact allows us to think that $h(t)$ can be deduced from the A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ and the sequence $(\rho_j)_{j\in\mathbb{N}}$. So, after having found the function $h(t)$, we can also find the A-sequence by determining its generating function $A(t)$. Almost always, $d_{n+1,k+1}$ only depends on the elements of a finite number of rows above it; therefore, instead of treating a global generating function for the A-matrix, let us examine a sequence of generating functions $P^{[0]}(t), P^{[1]}(t), P^{[2]}(t), \ldots$ corresponding to the rows $0, 1, 2, \ldots$ of the A-matrix, i.e.:

$$P^{[0]}(t) = \alpha_{0,0} + \alpha_{0,1}t + \alpha_{0,2}t^2 + \alpha_{0,3}t^3 + \ldots$$
$$P^{[1]}(t) = \alpha_{1,0} + \alpha_{1,1}t + \alpha_{1,2}t^2 + \alpha_{1,3}t^3 + \ldots$$

and so on. Moreover, let $Q(t)$ be the generating function for the sequence $(\rho_j)_{j\in\mathbb{N}}$. Thus we have:

Theorem 2.4. *If $(d_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array whose generic element $d_{n+1,k+1}$ is defined by formula* (2.6) *through the A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ and the sequence $(\rho_j)_{j\in\mathbb{N}}$,* *then the functions $h(t)$ and $A(t)$ for $(d_{n,k})$ are given by the following implicit expressions*:

$$\frac{h(t)}{t} = \sum_{i\geq 0} t^i P^{[i]}(h(t)) + \frac{h(t)^2}{t} Q(h(t)), \tag{2.7}$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i} P^{[i]}(t) + tA(t)Q(t). \tag{2.8}$$

The generic element $d_{n+1,k+1}$ often only depends on the two previous rows and sometimes on the elements of its own row. In this case, the functional equation (2.8) reduces to a second degree equation in $A(t)$ and, as a result, we give an explicit expression for the generating function of the A-sequence.

Theorem 2.5. *Let $(d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array whose generic element $d_{n+1,k+1}$ only depends on the two previous rows and, possibly, on its own row. If $P^{[0]}(t)$, $P^{[1]}(t)$ and $Q(t)$ are the generating functions for the coefficients of this dependence, then we have*:

$$A(t) = \frac{P^{[0]}(t) + \sqrt{P^{[0]}(t)^2 + 4tP^{[1]}(t)(1 - tQ(t))}}{2(1 - tQ(t))}. \tag{2.9}$$

As mentioned above, $h(t)$ is related to $A(t)$ and $d(t)$ is related to $Z(t)$. Since the Z-sequence exists for every lower triangular array, every recurrence defining $d_{n+1,0}$ in terms of the other elements in the array can be accepted as a good definition of column 0. Therefore, in analogy to (2.6), we have the following linear relation:

$$d_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0} \zeta_{i,j} d_{n-i,j} + \sum_{j\geq 0} \sigma_j d_{n+1,j+2}. \tag{2.10}$$

In general, there is no connection between the $\zeta_{i,j}$'s and the $\alpha_{i,j}$'s or between the

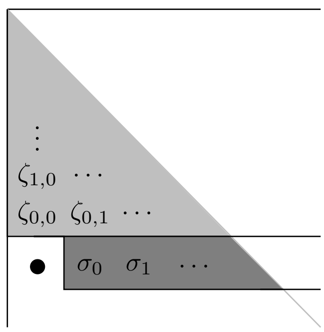


FIGURE 3. The zones which $d_{n+1,0}$ can depend on.

ρ_j's and the σ_j's and so we take the following generating functions into account:

$$R^{[0]}(t) = \zeta_{0,0} + \zeta_{0,1}t + \zeta_{0,2}t^2 + \zeta_{0,3}t^3 + \cdots$$
$$R^{[1]}(t) = \zeta_{1,0} + \zeta_{1,1}t + \zeta_{1,2}t^2 + \zeta_{1,3}t^3 + \cdots$$

and so on, and $S(t) = \sum_{j\geq 0}\sigma_j t^j$. When the coefficients defining $d_{n+1,k+1}$ and $d_{n+1,0}$ are the same, in the sense that:

$$\zeta_{i,j} = \alpha_{i,j+1} \text{ and } \sigma_j = \rho_j \ \forall i, \forall j,$$

we say that *column* 0 *is unprivileged* and obtain the following formulas for our generating functions:

$$R^{[i]}(t) = \frac{P^{[i]}(t) - \alpha_{i,0}}{t} \quad \text{and} \quad S(t) = Q(t)$$

for every i for which $R^{[i]}(t)$, $P^{[i]}(t)$, $S(t)$ and $Q(t)$ are well-defined.

At any rate, we can easily prove the following:

Theorem 2.6. *If* $(d_{n,k})_{n,k\in\mathbb{N}}$ *is a Riordan array whose elements in column 0 are defined by relation* (2.10)*, then the function* $d(t)$ *is given by the following formula*:

$$d(t) = \frac{d_{0,0}}{1 - \sum\limits_{i\geq 0} t^{i+1}R^{[i]}(h(t)) - h(t)S(h(t))}. \tag{2.11}$$

When column 0 is unprivileged, Theorem 2.6 reduces to $d(t) = (d_{0,0}h(t))/\sum\limits_{i\geq 0}\alpha_{i,0}t^{i+1}$, therefore, when $d_{0,0} = \alpha_{0,0}$ and $\alpha_{i,0} = 0$, $\forall i > 0$, we obtain the *Bell subgroup* whose elements are called *renewal arrays* (see, e.g., [**14**]).

It is clear that, in general, only proper Riordan arrays with integer coefficients can have a direct combinatorial interpretation. So, we restrict our attention to formal power series $f(t) = \sum\limits_{k=0}^{\infty} f_k t^k$ having $f_0 = 1$ and $f_k \in \mathbb{Z}$, for every $k \in \mathbb{N}$. These formal power series are called *monic, integer formal power series*. Consequently, we define *monic, integer proper Riordan arrays* as proper Riordan arrays whose elements are in $\mathbb{Z}$ and those on the main diagonal are 1. These arrays are the main object of our investigations.

3. LEVEL GENERATING TREES

In [**9**] Merlini, Sprugnoli and Verri extended the correspondence between generating trees and proper Riordan arrays to the whole group of monic integer proper Riordan arrays. The generating trees have been extended to deal with *marked* labels: a label is any positive integer, generated according to the generating tree specification; a marked label is any positive integer, marked by a bar, for which appropriate rules are given in the specification:

Definition 3.1. *A marked generating tree is a rooted labeled tree* (*the labels can be marked or non-marked*) *with the property that if* v_1 *and* v_2 *are any two nodes with the same label then, for each label* l, v_1 *and* v_2 *have exactly the same number of children with label* l. *To specify a generating tree it therefore suffices to specify*:

1. *the label of the root*;
2. *a set of rules explaining how to derive from the label of a parent the labels of all of its children.*

A simple example is given by the following generating tree specification:

$$\left\{\begin{array}{ll} root: & (2) \\ rule: & (k) \rightarrow (\overline{k})(k+1) \\ & (\overline{k}) \rightarrow (k)(\overline{k+1}) \end{array}\right. \tag{3.1}$$

The first 4 levels of the corresponding generating tree are shown in Figure 4.

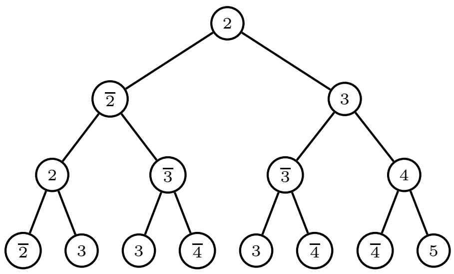


FIGURE 4. The generating tree for rule (3.1).

The idea is that marked labels kill or annihilate the non-marked labels with the same number, i.e. the count relative to an integer j is the difference between the number of non-marked and marked labels j at a given level. This gives a negative count if marked labels are more numerous than non-marked ones.

Definition 3.2. *An infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ is said to be* associated *to a marked generating tree with root (c) (AGT matrix for short) if $d_{n,k}$ is the difference between the number of nodes at level n with label $k+c$ and the number of nodes with label $\overline{k+c}$. By convention, the level of the root is 0.*

The triangle in Table 2 corresponds to the AGT matrix associated to the specification (3.1). We observe that the row sums of an AGT matrix can be simply

n/k	0	1	2	3	4
0	1				
1	-1	1			
2	1	-2	1		
3	-1	3	-3	1	
4	1	-4	6	-4	1

TABLE 2. The AGT matrix associated to (3.1)

evaluated by formula (2.1) with $f_k = 1$ and $f(t) = \dfrac{1}{1-t}$. Before stating the main result obtained by MERLINI et al. in [**9**], we introduce the following notations for generating tree specifications.

$$\begin{array}{rcl}
(x) & = & (\overline{\overline{x}}) \\
(x)^p & = & \underbrace{(x)\cdots(x)}_{p\ times},\ p \geq 0 \\
(x)^p & = & \underbrace{(\overline{x})\cdots(\overline{x})}_{-p\ times},\ p < 0 \\
\overline{(x)^p} & = & (\overline{x})^p,\ p > 0 \\
\overline{(x)^p} & = & (x)^{-p},\ p < 0 \\
\prod_{j=0}^{i}(k-j)^{\alpha_j} & = & (k)^{\alpha_0}(k-1)^{\alpha_1}\cdots(k-i)^{\alpha_i}
\end{array}$$

We note that $(x)^0$ is the empty sequence and finally state the following theorem which relates monic integer proper RIORDAN arrays to marked generating trees:

Theorem 3.1. *Let $c \in \mathbb{N}$, a_j, $b_k \in \mathbb{Z}$, $\forall j \geq 0$ and $k \geq c$, $a_0 = 1$, and let*

$$(3.2) \qquad \begin{cases} root: & (c) \\ rule: & (k) \to (c)^{b_k} \displaystyle\prod_{j=0}^{k+1-c}(k+1-j)^{a_j} \\ & (\overline{k}) \to \overline{(c)^{b_k}} \displaystyle\prod_{j=0}^{k+1-c} \overline{(k+1-j)^{a_j}} \end{cases}$$

be a marked generating tree specification. Then, the AGT matrix associated to (3.2) *is a monic integer proper Riordan array defined by the triple (d_0, A, Z), such that*

$$d_0 = 1,\ A = (a_0, a_1, a_2, \ldots),\ Z = (b_c + a_1, b_{c+1} + a_2, b_{c+2} + a_3, \ldots).$$

Vice versa, if D is a monic integer proper Riordan array defined by the triple $(1, A, Z)$ with $a_j, z_j \in \mathbb{Z}$, $\forall j \geq 0$ and $a_0 = 1$, then D is the AGT matrix associated to the generating tree specification (3.2) *with $b_{c+j} = z_j - a_{j+1}$, $\forall j \geq 0$.*

This can be generalized as follows:

Definition 3.3. *A Level Generating Tree* (*LGT*) *is a rooted labelled tree* (*the labels can be marked or non-marked*) *with the property that, if v_1 and v_2 are any two nodes at level i and j respectively with the same label then, for each label l and level n, v_1 and v_2 have exactly the same number of children at level $i+n$ and $j+n$ respectively, with label l. To specify a LGT it therefore suffices to specify*:

1. *the label of the root*;
2. *a set of rules explaining how to derive from the label of a parent the labels of all its children* (*different level*) *and brothers* (*same level*).

Definition 3.4. *An infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ is said to be "associated" to a LGT with root (c) (ALGT matrix for short) if $d_{n,k}$ is the difference between the number of nodes at level n with label $k+c$ and the number of nodes at level n with label $\overline{k+c}$. By convention, the level of the root is 0.*

Theorem 3.2. *Let $c \in \mathbb{N}$, $\alpha_{i,j} \in \mathbb{Z}$, $\alpha_{0,0} \neq 0$, $\rho_i \in \mathbb{Z}$, $\zeta_{i,j} \in \mathbb{Z}$, $\sigma_i \in \mathbb{Z}$, $\forall i,j \geq 0$ and $k \geq c$, and let*

$$
(3.3) \qquad \begin{cases}
\textit{root}: & (c) \\
\textit{rule}: & (k)_i \to (c)_{i+j+1}^{\zeta_{j,k}} \prod_{h=0}^{k-c} (k+1-h)_{i+j+1}^{\alpha_{j,h}} \\
& (\overline{k})_i \to \overline{(c)_{i+j+1}^{\zeta_{j,k}}} \prod_{h=0}^{k-c} \overline{(k+1-h)_{i+j+1}^{\alpha_{j,h}}} \\
& \dots \\
& (k)_i \to (c)_{i+1}^{\zeta_{0,k}} \prod_{h=0}^{k-c} (k+1-h)_{i+1}^{\alpha_{0,h}} \\
& (\overline{k})_i \to \overline{(c)_{i+1}^{\zeta_{0,k}}} \prod_{h=0}^{k-c} \overline{(k+1-h)_{i+1}^{\alpha_{0,h}}} \\
& (k)_i \to (c)_i^{\sigma_{k-1}} \prod_{h=1}^{k-c-1} (k-h)_i^{\rho_{h-1}} \\
& (\overline{k})_i \to \overline{(c)_i^{\sigma_{k-1}}} \prod_{h=1}^{k-c-1} \overline{(k-h)_i^{\rho_{h-1}}}
\end{cases}
$$

be a level generating tree specification. Then, the ALGT matrix associated to (3.3) *is a proper Riordan array D defined by the A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, Z-matrix $(\zeta_{i,j})_{i,j\in\mathbb{N}}$ and the sets of sequences $(\rho_i)_{i\in\mathbb{N}}$ and $(\sigma_i)_{i\in\mathbb{N}}$ with $d_{0,0} = 1$.*

Proof. Let us consider the ALGT matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ associated to (3.3). Then, $d_{n,k}$ counts the number of nodes at level n with label $k + c$. We have obviously $d_{0,0} = 1$. Moreover, we observe that the maximum label value at each level increases by one with respect to the previous level, hence $d_{n,j} = 0$ for $j > n$. Now, (3.3) tells us that:

- a node at level $n + 1$ with label $k + 1 + c$ can be determined, in $\alpha_{j,h}$ different ways, from the nodes at the level $n - j$ with label $z + c$, such that $z + c + 1 - h = k + c + 1$, i.e., $z = k + h$, $h \geq 0$;
- a node at level $n+1$ with label $k+1+c$ can be determined, in ρ_h different ways, from the nodes at the same level $n + 1$ with label $z + c$, such that $z + c - h = k + c + 1$, i.e., $z = k + h + 1$, $h \geq 1$.

Hence:

$$
\begin{aligned}
d_{n+1,k+1} = {} & \rho_0 d_{n+1,k+2} + \rho_1 d_{n+1,k+3} + \cdots + \\
& \alpha_{0,0} d_{n,k} + \alpha_{0,1} d_{n,k+1} + \cdots\cdots + \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
& \alpha_{j,0} d_{n-j,k} + \alpha_{j,1} d_{n-j,k+1} + \cdots .
\end{aligned}
$$

In a similar way, a node at level $n+1$ with label c can be obtained from the nodes at the same level and from the nodes at previous levels. Therefore we have:

$$\begin{aligned} d_{n+1,0} = \sigma_0 d_{n+1,1} + \sigma_1 d_{n+1,2} + \cdots \ + \\ \zeta_{0,0} d_{n,0} + \zeta_{0,1} d_{n,1} + \cdots\cdots + \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ \zeta_{j,0} d_{n-j,0} + \zeta_{j,1} d_{n-j,1} + \cdots . \end{aligned}$$

□

We call *proper Level Generating Trees* the level generating trees corresponding to Theorem 3.2.

EXAMPLE 3.1. The connection between the language of words avoiding a given pattern and RIORDAN arrays is of interest to us because the resulting matrices are better defined by means of an A-matrix rather than by an A-sequence. In [**1**] we have studied binary words excluding a pattern $\mathfrak{p} = p_0 \dots p_{h-1} \in \{0,1\}^h$ with respect to the number of zeroes and ones.

Let us consider the pattern $\mathfrak{p} = 11100$ and let us apply the method described in [**1**] to obtain the RIORDAN array $R_{n,j}$ counting the number of words of length $2n-j$ and with n bits equal to 1:

$n\backslash j$	0	1	2	3	4	5	6	…
0	1	0	0	0	0	0	0	…
1	2	1	0	0	0	0	0	…
2	6	3	1	0	0	0	0	…
3	18	9	4	1	0	0	0	…
4	58	29	13	5	1	0	0	…
5	192	96	44	18	6	1	0	…
6	650	325	151	64	24	7	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

In this case the basic recurrence for the RIORDAN array is:

$$R_{n+1,j+1} = R_{n,j} + R_{n+1,j+2} - R_{n-2,j}. \tag{3.4}$$

This shows that the combinatorial problem is described by an A-matrix, containing simple coefficients. In principle, this should be equivalent to some combinatorial proof relating the elements of row $n+1$, to elements in the same or in the previous rows. If we look for the A-sequence corresponding to our simple A-matrix, we find that

$$A(t) = 1 + t + 2t^3 - t^4 + 7t^5 - 12t^6 + 38t^7 - 99t^8 + \dots$$

and this excludes that there might exist a “simple” dependence of the elements in row $n+1$ from the elements in row n. In order to use the LGT we need the Z-matrix, that is:

$$R_{n+1,0} = 2R_{n+1,1} \tag{3.5}$$

Finally, from recurrences (3.4) and (3.5) we define the following rule:

$$
\left\{
\begin{array}{lll}
root: & (0) & \\
rule: & (1)_i & \to (0)_i^2 \\
 & (k)_i & \to (k-1)_i, \quad k>1 \\
 & (k)_i & \to (k+1)_{i+1} \\
 & (k)_i & \to (\overline{k+1})_{i+3}
\end{array}
\right.
\tag{3.6}
$$

Figure 5 shows the LGT for this problem together with the A-matrix and Z-matrix.

4. BLOOM'S STRINGS

BLOOM [**3**] studies the number of singles in all the 2^n n-length bit strings, where a *single* is any isolated 1 or 0, i.e., any run of length 1. Let $R_{n,k}$ be the number of n-length bit strings beginning with 0 and having k singles. We want to examine a generic n-length bit string $\beta_n = b_0 b_1 \cdots b_{n-1}$ with $b_0 = 0$ having k singles:

- $n = 0, k = 0$: we have the empty string having no singles ($R_{0,0} = 1$);
- $n = 1, k = 0$: there are no bit strings with no single ($R_{1,0} = 0$);
- $n > 1, k = 0$: the bit b_1 must be equal to 0 because the bit b_0 is not a single. We have two cases:
    1. $b_1 = 0$ is a single in $b_1 \cdots b_{n-1}$: the single b_1 is deleted by b_0, and the bit string $b_2 \cdots b_{n-1}$ has 0 singles. Therefore, we have $R_{n-2,0}$ configurations;
    2. $b_1 = 0$ is not a single in $b_1 \cdots b_{n-1}$: the bit string $b_1 \cdots b_{n-1}$ has 0 singles, and these are $R_{n-1,0}$ configurations.
- $n > 0, k > 0$: we examine the following cases
    1. $b_1 = 1$. In this case we can say that $b_1 \cdots b_{n-1}$ has only $k-1$ singles. Using the complementary strings, we have $R_{n-1,k-1}$ configurations.
    2. $b_1 = 0$. In this case we can say that $b_1 \cdots b_{n-1}$ has k singles. Moreover, we have two possibilities:
        - b_1 is a single in $b_1 \cdots b_{n-1}$;
        - b_1 is not a single in $b_1 \cdots b_{n-1}$.

       Therefore, we use $R_{n-1,k}$ in order to count the number of $(n-1)$-length bit strings beginning with 0 and having k singles and eliminate the cases in which the bit b_0 is a single with $R_{n-2,k-1}$. Finally, we use $R_{n-2,k}$ to consider the number of $(n-2)$-length bit strings beginning with 0 and having k singles.

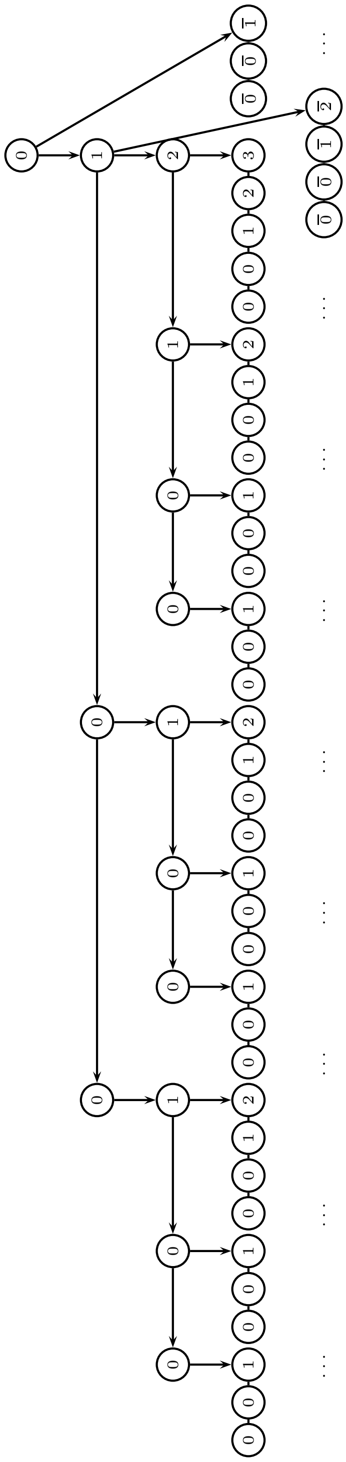
In this way, we have the following relations:

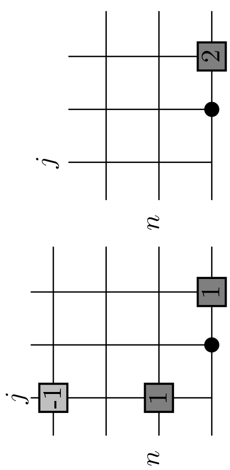
$$R_{0,0} = 1$$

$$R_{1,0} = 0$$

$$R_{n+2,0} = R_{n+1,0} + R_{n,0} \tag{4.1}$$

$$R_{n+1,k+1} = R_{n,k} + R_{n,k+1} - R_{n-1,k} + R_{n-1,k+1}. \tag{4.2}$$





(a) A-matrix

(a) Z-matrix

FIGURE 5. LGT of Binary words excluding the pattern $\mathfrak{p} = 11100$

Using Theorem 2.3, the resulting matrix $\mathbf{R} = (R_{n,k})$ is a Riordan array where each element differs from the corresponding element in Bloom's array by a factor of 2, apart from $n = 0$ and $k = 0$ (see Table 3). To find the $d(t)$ function

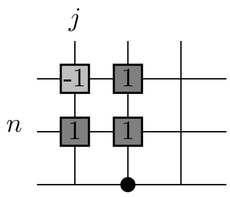
n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	0	1					
3	1	2	0	1				
4	2	2	3	0	1			
5	3	5	3	4	0	1		
6	5	8	9	4	5	0	1	
7	8	15	15	14	5	6	0	1

Table 3. The number of n length bit strings beginning with 0, having k singles.

(the generating function of column 0 in the matrix $\mathbf{R}$), we note from (4.1) that this is the generating function of the Fibonacci numbers with initial conditions $d_0 = 1$ and $d_1 = 0$. In this way, we obtain:

$$d(t) = \frac{1-t}{1-t-t^2}. \tag{4.3}$$

We can apply Theorem 2.4 to find $h(t)$ and the A-sequence as follows. In this case, using (4.2) we have an A-matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ which can be represented as follows:



This translates into the generating functions

$$P^{[1]}(t) = -1 + t, \quad P^{[0]}(t) = 1 + t.$$

By replacing $P^{[0]}(t)$ and $P^{[1]}(t)$ in formula (2.7) we obtain the generating function for $h(t)$:

$$\frac{h(t)}{t} = P^{[0]}\left(h(t)\right) + tP^{[1]}\left(h(t)\right),$$

thus finding:

$$h(t) = \frac{t-t^2}{1-t-t^2}. \tag{4.4}$$

By using generating functions, we can compute the number of singles in all the 2^{n-1} n-length bit strings beginning with 0:

$$R(t,w) = \frac{d(t)}{1-wh(t)} = \frac{1-t}{1-t(1+w)-t^2(1-w)}; \tag{4.5}$$

the coefficient $[t^n w^k]R(t,w)$ gives the number of n-length bit strings beginning with 0 and having k singles. The same function obviously counts the n-length bit strings beginning with 1 and having k singles.

For the A-sequence we use formula (2.9) and obtain

$$\begin{aligned} A(t) &= \frac{1+t+\sqrt{(1+t)^2+4t(-1+t)}}{2} = \frac{1+t+\sqrt{1-2t+5t^2}}{2} \\ &= 1+t^2+t^3-2t^5-3t^6+t^7+11t^8+15t^9+O(t^{10}). \end{aligned} \tag{4.6}$$

In this case we have found a quite *complex* A-sequence (an infinite sequence with increasing values, in modulus, and alternating signs) in contrast to a very simple A-matrix (see $P^{[0]}(t)$ and $P^{[1]}(t)$ before) and therefore the original concept of a generating tree cannot be naturally associated to this example. However, the new concept of LGT is appropriate for this case: the root has label 0 and produces a node with label 1 at level 1 and two nodes with labels $0, \overline{1}$ at level 2. In general, a node with label k at level i produces two nodes with labels $k, k+1$ at level $i+1$ and two nodes with labels $k, \overline{k+1}$ at level $i+2$. The specification rule becomes:

$$\begin{cases} root: & (0) \\ rule: & (0)_0 \to (1)_1 \\ & (0)_0 \to (0)_2(\overline{1})_2 \\ & (k)_i \to (k)_{i+1}(k+1)_{i+1} \qquad\qquad i>0 \\ & (k)_i \to (k)_{i+2}(\overline{k+1})_{i+2} \end{cases} \tag{4.7}$$

To this tree structure, we can easily associate a matrix in which every element counts the difference between non marked and marked nodes with the same label at a certain level: in this way we obtain exactly the matrix in Table 3. The next step, consists in using the level generating tree in order to generate the Bloom strings up to a certain length. We propose the following algorithm, where b_i and $\overline{b}_i$ represent a bit and its complement and the symbol $\rightsquigarrow$ is used to associate a node to the corresponding Bloom's string:

$\forall i$, let b_i a bit and $\overline{b}_i$ its complementary.
$(0)_0 \rightsquigarrow \epsilon$
$(1)_1 \rightsquigarrow 0$
$(0)_2 \rightsquigarrow 00$
$(\overline{1})_2 \rightsquigarrow 00$
For all nodes $(k)_i$:
  if $(k)_i \rightsquigarrow b_0 \cdots b_{i-1}$ then
    $(k)_{i+1} \rightsquigarrow b_0 \cdots b_{i-1} b_{i-1}$
    $(k+1)_{i+1} \rightsquigarrow b_0 \cdots b_{i-1} \overline{b}_{i-1}$
    $(k)_{i+2} \rightsquigarrow b_0 \cdots b_{i-1} \overline{b}_{i-1} \overline{b}_{i-1}$
    $(\overline{k+1})_{i+2} \rightsquigarrow b_0 \cdots b_{i-1} \overline{b}_{i-1} \overline{b}_{i-1}$

This algorithm associates to every node k at level i a Bloom's string of length i with k singles, as illustrated in Figure 6. We observe from Figure 6 that some

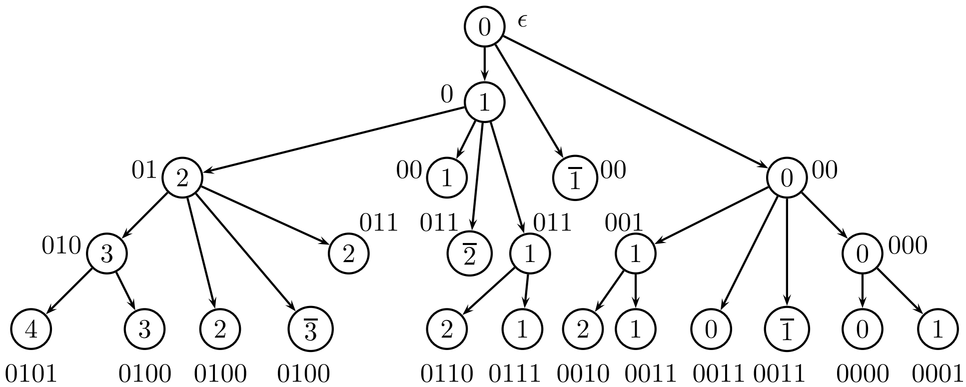
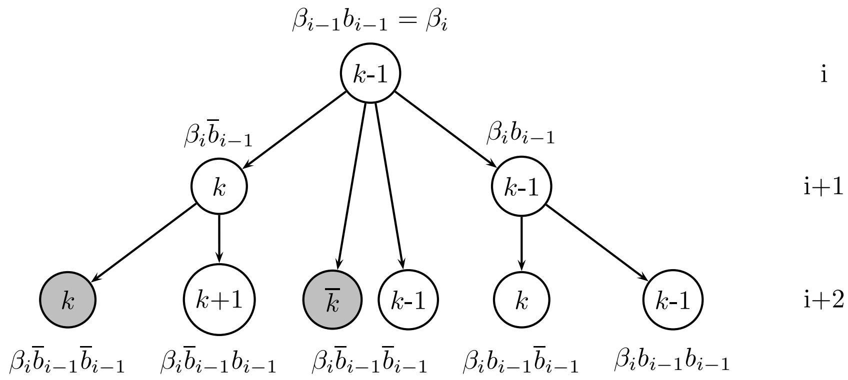


FIGURE 6. Generating Bloom's string with the LGT (4.7)

strings are generated several times: for example, string 0100 at level 4 corresponds to the nodes with label 2, 3 and $\overline{3}$. However, the string having the right association with its node (i.e. the string having a number of singles equal to the value of the label) is the one corresponding to label 2. The other two strings must be deleted during the process of generation. In general, every marked node $(\overline{k})_i$ deletes a node $(k)_i$ having the same label and the same bit string (which has only $k-1$ singles), as follows:



In fact, the algorithm assigns to $(k)_{i+1}$ the bit string $\beta_i\overline{b}_{i-1}$, where $\overline{b}_{i-1}$ is a single by construction. On the other side, the node $(k)_{i+1}$ generates the node $(k)_{i+2}$ with associated bit string $\beta_i\overline{b}_{i-1}\overline{b}_{i-1}$. This bit string has $k-1$ singles, and therefore its corresponding node $(k)_{i+2}$ must be deleted by $(\overline{k})_{i+2}$ Finally, if we consider a node $(0)_{i+1}$, the associated bit string $\beta_i b_{i-1}$ corresponds to a sequence of 0. In this case, the child node $(0)_{i+2}$ has a legal associated bit string.

5. LATTICE PATH PROBLEMS

In this section we use a model of lattice paths studied in [**8**]. A lattice path of m steps is a finite sequence $(s_1, \cdots, s_m)$ of ordered pairs $s_i = ((x_{i-1}, y_{i-1}), (x_i, y_i))$, with $1 \leq i \leq m$, of lattice points such that:

1. $x_0 = y_0 = 0$;
2. for $1 \leq i \leq m$, $x_i = x_{i-1} + \delta_i$, $y_i = y_{i-1} + \delta'_i$;
3. the pairs (δ_i, δ'_i), $1 \leq i \leq m$, are drawn from a set of permissible step templates; and
4. these permissible step templates obey some conditions on their occurrence.

We consider the paths starting at the origin $(0,0)$ and ending at (x_m, y_m).

We are going to examine some lattice path problems having templates in the class $T = \{(\delta,\delta')|\delta,\delta' \in \mathbb{N}, \delta+\delta' > 0\} \cup \{(\delta,\delta')|\delta < 0, \delta' > 0\}$. We denote a step template (δ,δ') having $\delta \geq 0$ by $e^{\delta}n^{\delta'}$, where e stands for *east* and n for *north*; a template is *steep* if $\delta \leq \delta'$ and is *shallow* if $\delta > \delta' + 1$; if $\delta = \delta' + 1$ the template will be called *almost steep*. A step template (δ,δ') having $\delta < 0$ will be denoted by $w^{|\delta|}n^{\delta'}$, where w stands for *west*. Figure 7 (a) illustrates the possible steps in $\mathbb{Z}^2$. By following the notation used in [**7**], we call *steep* those steps with $\delta \leq \delta'$ (the clear grey zone in Figure 7), *almost steep* those steps with $\delta = \delta'$ (the dark grey zone in Figure 7), and *shallow* the ones with $\delta > \delta' + 1$ (the white zone in Figure 7). The paths we want to describe are characterized by the pair (R_A, R_Δ) where:

- R_A is a set of steps of the kind $n^{\delta'}e^{\delta}$ with $\delta,\delta' \geq 0$ or $\delta < 0, \delta' > 0$: this set is used for steps not ending on the main diagonal;

- R_Δ is a set of *steep* steps and is used for steps which do end on the main diagonal.

Let R_S be the subset of R_A made up of steep steps only: if $R_S \neq R_\Delta$ we say that the paths have *privileged access to the main diagonal*.

We now denote by $d_{n,k}$ the number of paths corresponding to a given pair (R_A, R_Δ) and arriving to the point $(n, n-k)$[1] of the lattice. In [**7**] the following result is proved:

Theorem 5.1. *Let (R_A, R_Δ) be a lattice path problem and let $(d_{n,k})_{n,k\in\mathbb{N}}$ be its corresponding counting array. Then $(d_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if R_A is composed of both steep steps and at least one almost steep step and a number s exists such that for each step $n^{\delta'}w^{|\delta|}$ with $\delta < 0$ we have $\delta' < s$. Besides, $(d_{n,k})_{n,k\in\mathbb{N}}$ is proper if R_A contains the almost steep step e.*

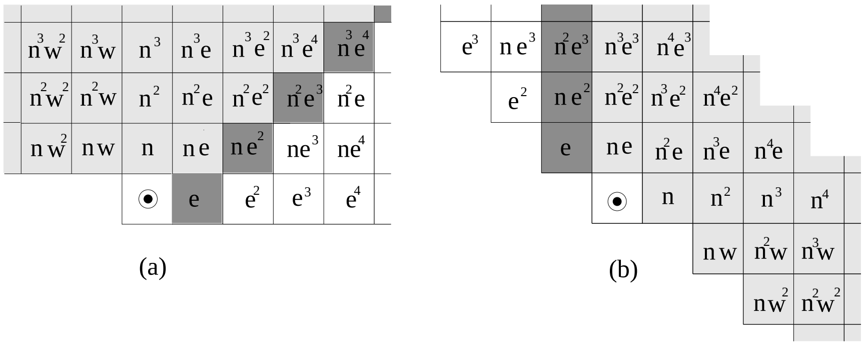


FIGURE 7. Possible steps originating from a given point in $\mathbb{Z}^2$ and their positions in the corresponding triangular array: e = *east*, n = *north*, w = *west*.

Figure 7 (b) illustrates, in terms of steps type, the dependence of the generic element $d_{n+1,k+1}$ from the other elements in the array. Since R_Δ is only made up of steep steps, the recurrence for $d_{n+1,0}$ does not depend on any elements in the white or dark-grey zone.

For example, the paths without privileged access to the main diagonal and $R_A = \{e, n, ne^2\}$ correspond to the RIORDAN array:

$$d_{n+1,k+1} = d_{n,k} + d_{n-1,k} + d_{n+1,k+2} \tag{5.1}$$

as shown by the results in [**7**]. In fact, in this case we have $P^{[0]}(t) = P^{[1]}(t) = Q^{[1]}(t) = 1$; therefore, $h(t)$ is given by the solution of $h(t) = t + t^2 + h(t)^2$; that is:

$$h(t) = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2}. \tag{5.2}$$

[1]We point out that we use the same symbol n for a north step and the abscissa of the arriving point of a path. The context allows to avoid any misunderstanding.

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	3	3	1				
3	9	9	5	1			
4	31	31	19	7	1		
5	113	113	73	33	9	1	
6	431	431	287	143	51	11	1

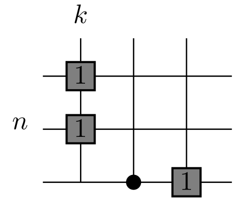


TABLE 4. The lower triangular array resulting from (5.1).

We should compute $d(t)$ by means of formula (2.11) in Theorem 2.6:

$$d(t) = \frac{1}{1-h(t)} = \frac{1-\sqrt{1-4t-4t^2}}{2t(1+t)}. \tag{5.3}$$

Finally, for the A-sequence we use the formula (2.9) and obtain

$$A(t) = \frac{1+\sqrt{1+4t(1-t)}}{2(1-t)} = 1 + 2t + 4t^3 - 8t^4 + 32t^5 - 112t^6 + 432t^7 \cdots$$

In this case, from recurrence (5.1) we define the following rule:

$$\left\{ \begin{array}{lll} root: & (0) & \\ rule: & (k)_i & \to (k+1)_{i+2} \\ & (k)_i & \to (k+1)_{i+1} \\ & (k)_i & \to (k-1)_i \end{array} \right. \tag{5.4}$$

Using Figure 7 (b) and (3.3), we propose the following algorithm for generating lattice paths according to our model:

$(0)_0 \rightsquigarrow \epsilon$
For all nodes $(k)_i$:
  if $(k)_i \rightsquigarrow Path$ then
    $(k+j)_i \rightsquigarrow Path \cdot n^j$
    $(k+j)_{i+h} \rightsquigarrow Path \cdot e^h n^{h-1+|j-1|}$

This algorithm associates to every LGT's node a path of the lattice. In particular, to every node $(k)_n$ we associate a path which arrives to the point $(n, n-k)$ of the lattice. For example, Figure 8 illustrates the paths corresponding to rule (5.4).

A referee observed that another lattice path problem for which the A-matrix is more convenient than the A-sequence is related to large SCHRÖDER numbers. In our model this problem corresponds to steps $R_A = \{e, n^2e, n^2e^2\}$ and no privileged

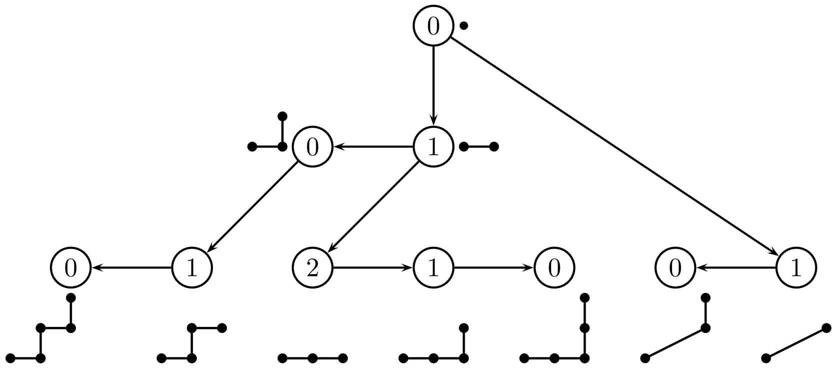


FIGURE 8. Generating lattice paths with the LGT corresponding to rule (5.4)

access to the main diagonal, that is, to the RIORDAN array:

$$d_{n+1,k+1} = d_{n,k} + d_{n-1,k+1} + d_{n,k+2}. \tag{5.5}$$

In terms of generating functions we have:

$$d(t) = \frac{1-t^2-\sqrt{1-6t^2+t^4}}{2t^2}, \quad h(t) = \frac{d(t)}{t}$$

and, for what concerns the A-sequence:

$$A(t) = \frac{1+t^2+\sqrt{1+6t^2+t^4}}{2} = 1+2t^2-2t^4+6t^6-22t^8+90t^{10}-394t^{12}+\cdots.$$

n/k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	2	0	1				
3	0	4	0	1			
4	6	0	6	0	1		
5	0	16	0	8	0	1	
6	22	0	30	0	10	0	1

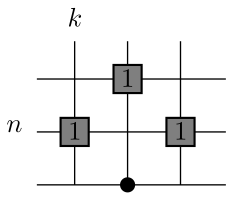


TABLE 5. The lower triangular array resulting from (5.5).

Using (5.5) we can define the following rule, which is illustrated in Figure 9:

$$\left\{ \begin{array}{lll} \mathit{root}: & (0) & \\ \mathit{rule}: & (k)_i & \to (k)_{i+2} \\ & (k)_i & \to (k+1)_{i+1} \\ & (k)_i & \to (k-1)_{i+1} \end{array} \right. \tag{5.6}$$

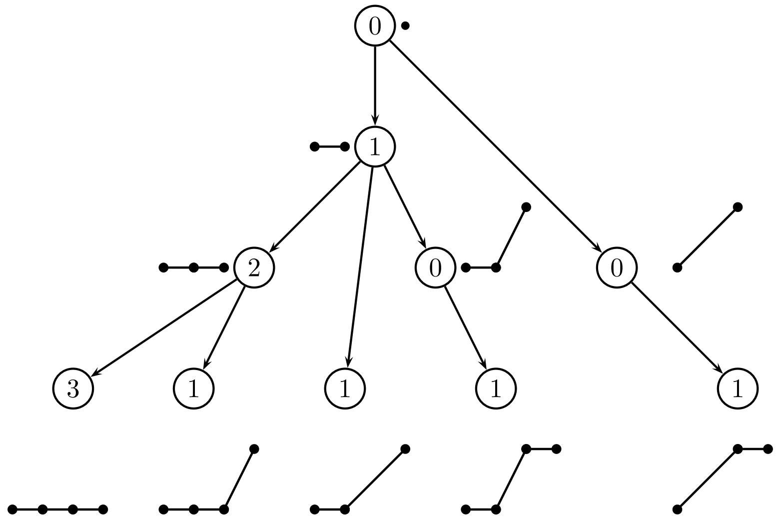


FIGURE 9. Generating lattice paths with the LGT corresponding to rule (5.6).

CONCLUSIONS

Our main result, which we tried to illustrated by several examples, allows us to associate the new concept of level generating trees to a class of RIORDAN arrays, better defined in terms of an A-matrix, rather than by their A-sequence. While the original concept of generating tree could only be used when combinatorial objects of size $n+1$ (say Ω_{n+1}) were defined by means of objects of size n, the new definition allows us to define Ω_{n+1} by objects in Ω_n, Ω_{n-1} and so on. Our examples show that this is a natural need for approaching many combinatorial problems (see also FERRARI et al. [**5**]), and therefore we consider this paper as a step towards a more general method to deal in a systematic way with enumeration problems in combinatorics.

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