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SOLVABILITY OF A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND-ORDER DIFFERENCE EQUATIONS

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New sufficient conditions for the existence of at least one solution of a class of two-point boundary value problems for second-order nonlinear difference equations are established.

1. INTRODUCTION

In this paper we study the following two-point boundary value problem for the second-order nonlinear difference equation

$$(1)\qquad \begin{cases} x(k+1) - ax(k) + bx(k-1) = f\big(k, x(k), x(k+1)\big), \quad k \in \overline{1, n-1}, \\ x(0) = A, \quad x(n) = B, \end{cases}$$

where $a, b, A, B \in \mathbb{R}$, $n \geq 2$ is an integer and f is a continuous, scalar-valued function.

In recent years there have been many papers devoted to the solvability of two-point boundary value problems for second-order or higher-order difference equations. We refer the reader to the textbooks [**1,2,3**], the papers [**6,7**] and the references therein.

In [**4**] the following discrete boundary value problem (BVP) involving the second-order difference equations and two-point boundary conditions:

$$(2)\qquad \begin{cases} \dfrac{\nabla \triangle y_k}{h^2} = f\left(t_k, y_k, \dfrac{\triangle y_k}{h}\right), \ k \in \overline{1, n-1}, \\ y_0 = A, y_n = B, \end{cases}$$

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is studied, where $A, B \in \mathbb{R}$, $n \geq 2$ is an integer, f is a continuous, scalar-valued function, the step size is $h = N/n$ with N a positive constant and the grid points are $t_k = kh$ for $k \in \overline{0, n}$. The differences are given by

$$\triangle y_k = \begin{cases} y_{k+1} - y_k, & k \in \overline{0, n-1}, \\ 0, & k = n; \end{cases}$$

$$\triangledown \triangle y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & k \in \overline{1, n-1}, \\ 0, & k = 0 \text{ or } k = n. \end{cases}$$

The following two results were proved in [**4**].

Theorem RT1. *Let f be continuous on $[0, N] \times \mathbb{R}^2$ and α, β and K be nonnegative constants. If there exist $c, d \in [0, 1)$ such that*

$$|f(t, u, v)| \leq \alpha |u|^c + \beta |v|^d + K, (t, u, v) \in [0, N] \times \mathbb{R}^2, \tag{3}$$

then the discrete BVP(2) *has at least one solution.*

Theorem RT2. *Let f be continuous on $[0, N] \times \mathbb{R}^2$ and α, β and K be nonnegative constants. If*

$$|f(t, u, v)| \leq \alpha |u| + \beta |v| + K, (t, u, v) \in [0, N] \times \mathbb{R}^2 \tag{4}$$

and

$$\frac{\alpha N^2}{8} + \frac{\beta N}{2} < 1, \tag{5}$$

then the discrete BVP(2) *has at least one solution.*

In Theorems RT1 and RT2 assumptions (3) and (4) allow f to grow either sublinearly or linearly. A question appears naturally:

• Under what conditions does the discrete BVP(2) have at least one solution when f grows superlinearly?

BVP(2) is a special case of BVP(1) when $a = 2$ and $b = 1$. The purpose of this paper is to establish sufficient conditions guaranteeing the existence of solutions of BVP(1) under the assumption that f grows superlinearly. The corollary when the theorem obtained is applied to BVP(2) is new since we allow f to grow superlinearly. The question mentioned above is solved.

This paper is organized as follows. In Section 2 the main result of this paper is given. We give an example to illustrate the main result in Section 3.

2. MAIN RESULT

In this Section we establish the main result in this paper. The following abstract existence theorem is used in the proof of the main result. Its proof can be seen in [**5**].

Lemma 2.1. *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with* $\operatorname{Ker} L = \{0\}$, *$N : X \to Y$ is L-compact on each open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$, then there exists at least one $x \in \Omega$ such that $Lx = Nx$.*

Let $X = \mathbb{R}^{n+1} = Y$ be endowed with the norm $\|x\| = \max_{n \in \overline{0,n}} |x(n)|$. It is easy to see that X is a Banach space. Choose $D(L) = X$. Let

$$L : X \to Y, \ \ L\,x(k) = \begin{pmatrix} x(k+1) - ax(k) + bx(k-1) \\ x(0) \\ x(n) \end{pmatrix}, \ \ x \in D(L),$$

and $N : X \to Y$ by

$$N\,x(k) = \begin{pmatrix} f(k, x(k), x(k+1)) \\ A \\ B \end{pmatrix}.$$

Suppose

(B) the following problem

$$x(k+1) - ax(k) + bx(k-1) = 0, \ \ k \in \overline{1, n-1}, \ x(0) = 0 = x(n)$$

has the unique solution $x(k) = 0$ for all $k \in \overline{0,n}$.

It is easy to show that

(i). $x \in D(L)$ is a solution of $L(x) = N(x)$ implies that x is a solution of BVP(1).

(ii). $\operatorname{Ker} L = \{0\}$.

(iii). L is a Fredholm operator of index zero and N is $L-$compact on each open bounded subset of X.

Remark. If $a = 2$ and $b = 1$, then

$$x(k+1) - 2x(k) + x(k-1) = 0, \ \ k \in \overline{1, n-1}, \ x(0) = 0 = x(n)$$

has the unique solution $x(k) = 0$ for all $k \in [0, n]$. In fact let $A_0 = a, A_1 = aA_0 - b$ and $A_k = aA_{k-1} - bA_{k-2}$ for $k \in \overline{2, n-2}$. If $A_{n-2} \neq 0$, it is easy to show that

$$x(k+1) - ax(k) + bx(k-1) = 0, \ \ k \in \overline{1, n-1}, \ x(0) = 0 = x(n),$$

has the unique solution $x = 0$.

Theorem L. *Suppose that $b \geq 0$, $b \leq a - 1/2$ and (B) holds. Furthermore there exist numbers $\beta > 0$, $\theta \geq 1$, nonnegative sequences $p(n), q(n), r(n)$, functions $g(n,x,y)$, $h(n,x,y)$ such that $f(n,x,y) = g(n,x,y) + h(n,x,y)$,*

$$g(n,x,y)x \geq \beta |x|^{\theta+1}$$

and

$$|h(n,x,y)| \le p(n)|x|^{\theta} + q(n)|y|^{\theta} + r(n),$$

for all $n \in \{1,\dots,n-1\}$, $(x,y) \in \mathbb{R}^2$. *Then* BVP(1) *has at least one solution if*

$$\|p\| + \|q\| < \beta. \tag{6}$$

Proof. To apply Lemma 2.1 we consider $Lx = \lambda Nx$ for $\lambda \in [0,1]$. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in [0,1]\}$. For $x \in \Omega_1$ we have

$$\begin{cases} x(k+1) - ax(k) + bx(k-1) = \lambda f(k, x(k), x(k+1)),\ k \in \overline{1, n-1}, \\ x(0) = \lambda A, \\ x(n) = \lambda B. \end{cases} \tag{7}$$

So

$$\big(x(k+1) - ax(k) + bx(k-1)\big)x(k) = \lambda f\big(k, x(k), x(k+1)\big)x(k),\ k \in \overline{1, n-1}. \tag{8}$$

It is easy to see that

$$\begin{aligned} &2\sum_{k=1}^{n-1}\big(x(k+1) - ax(k) + bx(k-1)\big)x(k) \\ &= \lambda^2 B^2 + b\lambda^2 A^2 + \sum_{k=1}^{n-1}\Big(-\big(x(k+1) - x(k)\big)^2 - b\big(x(k-1) - x(k)\big)^2\Big) \\ &+ \Big((b+2-2a)\big(x(n-1)\big)^2 + (2b+2-2a)\sum_{k=2}^{n-2}\big(x(k)\big)^2 + (2b+1-2a)\big(x(1)\big)^2\Big). \end{aligned}$$

Since $b \ge 0$ and $b \le a - 1/2$, we get $b \ge 0$, $b + 2 - 2a \le 0, 2b + 2 - 2a \le 0$ and $2b + 1 - 2a \le 0$. So we get

$$2\sum_{k=1}^{n-1}\big(x(k+1) - ax(k) + bx(k-1)\big)\,x(k) \le \lambda^2 B^2 + b\lambda^2 A^2.$$

Hence

$$\lambda \sum_{k=1}^{n-1} f\big(k, x(k), x(k+1)\big)x(k) \le \frac{1}{2}\,(\lambda^2 B^2 + b\lambda^2 A^2).$$

Then

$$\sum_{k=1}^{n-1}\Big(g\big(k, x(k), x(k+1)\big) + h\big(k, x(k), x(k+1)\big)\Big)x(k) \le \frac{1}{2}\,\big(B^2 + |b|A^2\big).$$

It follows that

$$\begin{aligned}
&\beta \sum_{k=1}^{n-1} |x(k)|^{\theta+1} \\
&\le \sum_{n=1}^{n-1} \big|h\big(k, x(k), x(k+1)\big)\big|\, |x(k)| + \frac{1}{2}\left(B^2 + |b|A^2\right) \\
&\le \sum_{n=1}^{n-1} \Big(p(k)|x(k)|^{\theta+1} + q(k)|x(k+1)|^{\theta}\, |x(k)| + r(k)|x(k)|\Big) + \frac{1}{2}\left(B^2 + |b|A^2\right) \\
&\le \|p\| \sum_{n=1}^{n-1} |x(k)|^{\theta+1} + \|q\| \sum_{k=1}^{n-1} |x(k+1)|^{\theta}\, |x(k)| + \|r\| \sum_{k=1}^{n-1} |x(k)| + \frac{1}{2}\left(B^2 + |b|A^2\right).
\end{aligned}$$

For $x_i \ge 0, y_i \ge 0$ HÖLDER′s inequality implies

$$\sum_{i=1}^{s} x_i y_i \le \left(\sum_{i=1}^{s} {x_i}^p\right)^{1/p} \left(\sum_{i=1}^{s} {y_i}^q\right)^{1/q}, \;\; 1/p + 1/q = 1, \;\; q > 0, \; p > 0.$$

It follows that

$$\begin{aligned}
&\beta \sum_{k=1}^{n-1} |x(k)|^{\theta+1} \\
&\quad \le \|p\| \sum_{n=1}^{n-1} |x(k)|^{\theta+1} + \|q\| \left(\sum_{k=1}^{n-1} |x(k+1)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{n-1} |x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}} \\
&\qquad + \|r\| \sum_{k=1}^{n-1} |x(k)| + \frac{1}{2}\big(B^2 + |b|A^2\big) \\
&\quad \le \|r\|(n-1)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{n-1} |x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}} + \|p\| \sum_{n=1}^{n-1} |x(k)|^{\theta+1} + \frac{1}{2}\big(B^2 + |b|A^2\big) \\
&\qquad + \|q\| \left(|A|^{\theta+1} + \sum_{k=1}^{n-1} |x(k)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}} \left(\sum_{k=1}^{n-1} |x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}.
\end{aligned}$$

We claim that, if $n > 0$, then there exists a constant $\sigma \in (0, 1)$, independent of λ, such that $(1 + x)^n \le 1 + (n + 1)x$ for all $x \in (0, \sigma)$. We consider two cases.

Case 1. $\sum_{k=1}^{n-1} |x(k)|^{\theta+1} \le \frac{|A|^{\theta+1}}{\sigma}$.

In this case we have

$$\sum_{k=1}^{n-1} |x(k)|^{\theta+1} \le \frac{|A|^{\theta+1}}{\sigma} =: M_1.$$

Case 2. $\sum\limits_{k=1}^{n-1}|x(k)|^{\theta+1} > \dfrac{A^{\theta+1}}{\sigma}$. Then $0 \le \dfrac{|A|^{\theta+1}}{\sum\limits_{k=1}^{n-1}|x(k)|^{\theta+1}} < \sigma$.

In this case one sees that

$$\begin{aligned}\beta\sum_{k=1}^{n-1}|x(k)|^{\theta+1} &\le \|r\|(n-1)^{\frac{\theta}{\theta+1}}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\ &\quad+\|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}+\frac{1}{2}\big(B^2+|b|A^2\big)\\ &\quad+\|q\|\left(\frac{A^{\theta+1}}{\sum\limits_{k=1}^{n-1}|x(k)|^{\theta+1}}+1\right)^{\frac{\theta}{\theta+1}}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}+\frac{1}{\theta+1}}\\ &\le \|r\|(n-1)^{\frac{\theta}{\theta+1}}\left(\sum_{k=1}^{n-1}|x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\ &\quad+\|p\|\sum_{n=1}^{n-1}|x(k)|^{\theta+1}+\frac{1}{2}\big(B^2+|b|A^2\big)\\ &\quad+\|q\|\sum_{k=1}^{n-1}|x(k)|^{\theta+1}+\|q\|\left(1+\frac{\theta}{\theta+1}\right)A^{\theta+1}.\end{aligned}$$

It follows from (6) that there exists a constant $M_2 > 0$ such that

$$\sum_{k=1}^{n-1}|x(k)|^{\theta+1} \le M_2.$$

Hence $|x(k)| \le \big(M/(n-1)\big)^{1/(\theta+1)}$ for all $k \in \overline{1, n-1}$, where $M = \max\{M_1, M_2\}$. Hence $\|x\| \le (M/(n-1))^{1/(\theta+1)}$. So Ω_1 is bounded.

Since f is continuous and (B) implies that $\operatorname{Ker} L = \{0\}$, we know that L is a Fredholm operator of index zero and N is $L-$compact on each open bounded subset of X. Let $\Omega_0 = \{x \in X : \|x\| < M+1\}$. Then $Lx \ne \lambda Nx$ for all $\lambda \in [0,1]$ and all $x \in D(L) \bigcap \partial\Omega_0$. It follows from Lemma 2.1 that BVP (1) has at least one solution. The proof is complete.

3. AN EXAMPLE

In this Section we present an example to illustrate the main result in Section 2.

EXAMPLE 3.1. Consider the following problem

$$(9)\qquad \begin{cases} \nabla \triangle x(k) = \beta\big(x(k)\big)^{2m+1} + p(k)\big(x(k)\big)^{2m+1} \\ \qquad\qquad + q(k)\big(x(k+1)\big)^{2m+1} + r(k), k \in \overline{1, n-1}, \\ x(0) = A, \\ x(n) = B, \end{cases}$$

where $n \geq 2$ is a positive integer, $m \geq 0$ an integer, $\beta > 0$, $p(n), q(n), r(n)$ are sequences and $\nabla \triangle x(k) = x(k+1) - 2x(k) + x(k-1)$. Corresponding to BVP(1) we set

$$\begin{aligned} f(k, x, y) &= \beta x^{2m+1} + p(k)x^{2m+1} + q(k)y^{2m+1} + r(k), \\ g(k, x, y) &= \beta x^{2m+1} \end{aligned}$$

and

$$h(k, x, y) = p(k)x^{2m+1} + q(k)y^{2m+1} + r(k).$$

It is easy to see that the assumptions in Theorem L hold. It follows from Theorem L that (9) has at least one solution if

$$\|p\| + \|q\| < \beta.$$

It is easy to see that BVP(9) cannot be solved by Theorems RT1 and RT2.

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