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A NOTE ON SIGNED DEGREE SETS IN SIGNED BIPARTITE GRAPHS

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A signed bipartite graph $G(U, V)$ is a bipartite graph in which each edge is assigned a positive or a negative sign. The signed degree of a vertex x in $G(U, V)$ is the number of positive edges incident with x less the number of negative edges incident with x. The set S of distinct signed degrees of the vertices of $G(U, V)$ is called its signed degree set. In this paper, we prove that every set of integers is the signed degree set of some connected signed bipartite graph.

1. INTRODUCTION

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [**3**]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The signed degree of a vertex v_i in G is denoted by $\mathrm{sdeg}(v_i)$ (or simply by d_{v_i} or by d_i) and is defined as $d_i = d_i^+ - d_i^-$, where $1 \le i \le n$ and $d_i^+ (d_i^-)$ is the number of positive (negative) edges incident with v_i. A signed degree sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order.

The various characterizations of signed degree sequences in signed graphs can be found in [**1,7**], and one such criterion [**1**] is similar to Hakimi's result for degree sequences in graphs [**2**].

The set of distinct signed degrees of the vertices of a signed graph is called its signed degree set. In [4], Kapoor et al. proved that every set of positive integers is the degree set of some connected graph and determined the smallest order for such a graph. Pirzada et al. [**6**] proved that every set of positive (negative) integers is the signed degree set of some connected signed graph and determined the smallest possible order for such a signed graph.

A graph G is called bipartite if its vertex set can be partitioned into two nonempty disjoint subsets U and V such that each edge in G joins a vertex in U

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with a vertex in V and is denoted by $G(U,V)$. Let $G(U,V)$ be a bipartite graph with $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$. Then degree of $u_i(v_j)$ is the number of edges of $G(U,V)$ incident with $u_i(v_j)$. The set of distinct degrees of the vertices of a bipartite graph $G(U,V)$ is called its degree set. Pirzada et al. [**5**] proved that every set of non-negative integers is a degree set of some bipartite graph.

2. MAIN RESULTS

A signed bipartite graph is a bipartite graph in which each edge is assigned a positive or a negative sign. Let $G(U,V)$ be a signed bipartite graph with $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$. The signed degree of u_i is $d_{u_i} = d_i = d_i^+ - d_i^-$, where $1 \le i \le p$ and d_i^+ (d_i^-) is the number of positive (negative) edges incident with u_i and signed degree of v_j is $d_{v_j} = e_j = e_j^+ - e_j^-$, where $1 \le j \le q$ and $e_j^+(e_j^-)$ is the number of positive (negative) edges incident with v_j. Clearly $|d_i| \le q$ and $|e_j| \le p$. The sequences $\alpha = [d_1, d_2, \ldots, d_p]$ and $\beta = [e_1, e_2, \ldots, e_q]$ are called the signed degree sequences of the signed bipartite graph $G(U,V)$. A signed bipartite graph $G(U,V)$ is said to be connected if each vertex $u \in U$ is connected to every vertex $v \in V$ by a path. For any two disjoint sets of vertices X and Y, we denote by $X \oplus Y$ to mean that each vertex of X is joined to every vertex of Y by a positive edge. The set S of distinct signed degrees of the vertices of a signed bipartite graph $G(U,V)$ is called its signed degree set.

The following result implies that every set of positive integers is a signed degree set of some connected signed bipartite graph.

Theorem 1.*Let $d_1, d_2, \ldots, d_n$ be positive integers. Then there exists a connected signed bipartite graph with signed degree set* $S = \left\{d_1, \sum_{i=1}^{2} d_i, \ldots, \sum_{i=1}^{n} d_i\right\}$.

Proof. If $n = 1$, then a signed bipartite graph $G(U,V)$ with $|U| = |V| = d_1$ and $U \oplus V$ has signed degree set $S = \{d_1\}$. For $n \ge 2$, construct a signed bipartite graph $G(U,V)$ as follows.

Let $U = X_1 \cup X_2 \cup X_2' \cup \cdots \cup X_n \cup X_n'$, $V = Y_1 \cup Y_2 \cup Y_2' \cup \cdots \cup Y_n \cup Y_n'$, with $X_i \cap X_j = \phi$, $X_i \cap X_j' = \phi$, $X_i' \cap X_j' = \phi$, $Y_i \cap Y_j = \phi$, $Y_i \cap Y_j' = \phi$, $Y_i' \cap Y_j' = \phi (i \ne j)$, $|X_i| = |Y_i| = d_i$ for all i, $1 \le i \le n$, $|X_i'| = |Y_i'| = d_1 + d_2 + \cdots + d_{i-1}$ for all i, $2 \le i \le n$. Let (i) $X_i \oplus Y_j$ whenever $i \ge j$, (ii) $X_i' \oplus Y_i$ for all i, $2 \le i \le n$ and (iii) $X_i' \oplus Y_i'$ for all i, $2 \le i \le n$. Then the signed degrees of the vertices of $G(U,V)$ are as follows.

For $1 \le i \le n$, $d_{x_i} = \sum_{j=1}^{i} |Y_j| = \sum_{j=1}^{i} d_j = d_1 + d_2 + \cdots + d_i$, for all $x_i \in X_i$; for $2 \le i \le n$, $d_{x_i'} = |Y_i| + |Y_i'| = d_i + d_1 + d_2 + \cdots + d_{i-1} = d_1 + d_2 + \cdots + d_i$, for all $x_i' \in X_i'$; for $1 \le i \le n$, $d_{y_i} = \sum_{j=i}^{n} |X_j| + |X_i'| = \sum_{j=i}^{n} d_j + d_1 + d_2 + \cdots + d_{i-1} = d_i + d_{i+1} + \cdots + d_n + d_1 + d_2 + \cdots + d_{i-1} = d_1 + d_2 + \cdots + d_n$, for all $y_i \in Y_i$; and for $2 \le i \le n$, $d_{y_i'} = |X_i'| = d_1 + d_2 + \cdots + d_{i-1}$, for all $y_i' \in Y_i'$.

Therefore signed degree set of $G(U,V)$ is $S=\left\{d_1,\sum_{i=1}^{2}d_i,\dots,\sum_{i=1}^{n}d_i\right\}$. Clearly by construction, all the signed bipartite graphs are connected. Hence the result follows.

By interchanging positive edges with negative edges in Theorem 1, we obtain the following result.

Corollary 2. *Every set of negative integers is a signed degree set of some connected signed bipartite graph.*

Finally we have the following result.

Theorem 3. *Every set of integers is a signed degree set of some connected signed bipartite graph.*

Proof. Let S, Z^+ and Z^- respectively be the set of integers, positive and negative integers. Then we have the following five cases.

(i)$S\subset Z^+(Z^-)$. The result follows by Theorem 1(Corollary 2).

(ii) $S=\{0\}$. Therefore a signed bipartite graph $G(U,V)$ with $|U|=|V|=2$ in which u_1v_1, u_2v_2 are positive edges and u_1v_2, u_2v_1 are negative edges, where $u_1,u_2\in U$, $v_1,v_2\in V$, has signed degree set S.

(iii) Let $S=S_1\cup\{0\}$, where $S_1\subset Z^+(Z^-)$, $S_1\neq\emptyset$. Then by Theorem 1(Corollary 2), there is a connected signed bipartite graph $G_1(U_1,V_1)$ with signed degree set S_1. Construct a new signed bipartite graph $G(U,V)$ as follows.

Let $U=U_1\cup\{x_1\}\cup\{x_2\}$, $V=V_1\cup\{y_1\}\cup\{y_2\}$, with $U_1\cap\{x_i\}=\phi,\{x_1\}\cap\{x_2\}=\phi$, $V_1\cap\{y_i\}=\phi,\{y_1\}\cap\{y_2\}=\phi$. Let u_1y_1, x_1v_1, x_2y_2 be positive edges and u_1y_2, x_1y_1, x_2v_1 be negative edges, where $u_1\in U_1$ and $v_1\in V_1$. Then $G(U,V)$ has degree set S. We note that addition of such edges do not affect the signed degrees of the vertices of $G_1(U_1,\,V_1)$, and the vertices x_1, x_2, y_1, y_2 have signed degrees zero each.

(iv) Let $S=S_1\cup S_2$, where $S_1\subset Z^+$, $S_2\subset Z^-$ and $S_1,S_2\neq\emptyset$. So by Theorem 1 and Corollary 2, there are connected signed bipartite graphs $G_1(U_1,\,V_1)$ and $G_2(U_2,\,V_2)$ with signed degree sets S_1 and S_2 respectively. Let $G_1'(U_1',\,V_1')$ and $G_2'(U_2',\,V_2')$ be the copies of $G_1(U_1,\,V_1)$ and $G_2(U_2,\,V_2)$ with signed degree sets S_1 and S_2 respectively. Construct a new signed bipartite graph $G(U,V)$ as follows.

Let $U=U_1\cup U_1'\cup U_2\cup U_2'$, $V=V_1\cup V_1'\cup V_2\cup V_2'$, with $U_i\cap U_j'=\phi$, $U_1\cap U_2=\phi$, $U_1'\cap U_2'=\phi$, $V_i\cap V_j'=\phi$, $V_1\cap V_2=\phi$, $V_1'\cap V_2'=\phi$. Let u_1v_2', $u_1'v_2$ be positive edges and u_1v_2, $u_1'v_2'$ be negative edges, where $u_i\in U_i$, $v_i\in V_i$, $u_i'\in U_i'$ and $v_i'\in V_i'$. Then $G(U,V)$ has signed degree set S. We note that addition of such edges do not affect the signed degrees of the vertices of $G_1(U_1,\,V_1)$, $G_1'(U_1',\,V_1')$, $G_2(U_2,\,V_2)$ and $G_2'(U_2',V_2')$.

(v) Let $S=S_1\cup S_2\cup\{0\}$, where $S_1\subset Z^+$, $S_2\subset Z^-$ and $S_1,S_2\neq\emptyset$. Then by Theorem 1 and Corollary 2, there exist connected signed bipartite graphs $G_1(U_1,\,V_1)$ and $G_2(U_2,\,V_2)$ with signed degree sets S_1 and S_2 respectively.

Construct a new signed bipartite graph $G(U,V)$ as follows. Let $U = U_1 \cup U_2 \cup \{x\}$, $V = V_1 \cup V_2 \cup \{y\}$, with $U_1 \cap U_2 = \phi$, $U_i \cap \{x\} = \phi$, $V_1 \cap V_2 = \phi$, $V_i \cap \{y\} = \phi$. Let u_1v_2, u_2y, xv_1 be positive edges and u_1y, u_2v_1, xv_2 be negative edges, where $u_i \in U_i$ and $v_i \in V_i$. Then $G(U,V)$ has signed degree set S. We note that addition of such edges do not affect the signed degrees of the vertices of $G_1(U_1,V_1)$ and $G_2(U_2,V_2)$, and the vertices x and y have signed degrees zero each.

Clearly by construction, all the signed bipartite graphs are connected. This proves the result.

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